Adaptive Investment Strategies for Different Scenarios

DISSERTATION

zur Erlangung des akademischen Grades

doctor rerum naturalium (Dr. rer. nat.)
im Fach Informatik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät II
Humboldt-Universität zu Berlin

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eingereicht am: 14. Juli 2008
Tag der mündlichen Prüfung: 6. Oktober 2008

This thesis is dedicated to
my beloved parents and my wonderful sisters.
Gracias por su paciencia, apoyo y amor.
Gott segne euch!!!

Abstract

The main goal of this PhD thesis is to investigate some of the problems related to optimization of resources in environments with unpredictable behavior where: (i) not all information is available and (ii) the environment presents unknown temporal changes. The investigations in this PhD thesis are divided in two parts: Part I presents the investment model and some analytical as well as numerical analysis of the dynamics of this model for fixed investment strategies in different random environments. In this investment model, the dynamics of the investor's budget $x(t)$ depend on the stochasticity of the exogenous return on investment $r(t)$ for which different model assumptions are discussed. The fat-tail distribution of the budget is investigated numerically and compared with theoretical predictions. Â Furthermore, it is shown that the most probable value x_{mp} of the budget reaches a constant value over time. Using simulations, the influence of the stochastic factors on the stationary most probable budget value is investigated. The results of these investigations suggest the presence of a scaling function between x_{mp} and the parameters characterizing the stochastic dynamics. The simulation results are corroborated by obtaining the scaling function analytically. Finally, the evolution of the budget of the agent is investigated for real returns from stock market data for different fixed proportions of investment and incomes. Part II investigates an investment scenario with stylized exogenous returns characterized by a periodic function with different types and levels of noise. In this scenario, different strategies, agent's behaviors and agent's capacities to predict the future $r(t)$ are investigated. Here, 'zero-intelligent' agents using technical analysis (such as moving least squares) are compared with agents using genetic algorithms to predict $r(t)$. The performance of an agent is measured by its average budget growth after a certain number of time steps. In order to ensure fair comparison between the strategies, the respective parameters of each of these strategies are adjusted so that they lead to maximal gains. Results are presented for extensive computer simulations, which shows that for exogenous returns with periodicity: (i) the daring behavior outperforms the cautious behavior and (ii) the genetic algorithm is able to find the optimal investment strategy by itself, thus outperforming the other strategies considered. These investigations are extended to find the best investment strategy for returns with changing periodicity. For this, the complexity of the strategy based on a Genetic Algorithm *GA* is extended by allowing the chromosomes to have a different length and considering more complex cross-over and mutation operators. In this way, the algorithm may find the correct mapping of proportions of investment to patterns that may be present in the returns. The performance of this adaptive investment strategy is compared with the performance of other investment strategies that were used as a reference. It is shown that after a number of time steps, the adaptive strategy reaches a set of investment strategies that can outperform simple strategies like those that always invest a constant proportion. Furthermore, it is shown that even though the adaptive strategy has no knowledge of the dynamics of the returns, it may lead to large gains, performing as well as other strategies with some knowledge. Finally, the investment model is extended to include the formation of common investment projects between agents. In this scenario, each project is conducted by an initiator who tries to convince other agents to invest in his project. An agent's decision to invest depends on its previous experience with the particular initiator. The influence of the parameters on the dynamics of the budget and the dynamics of the investment networks are investigated. It is shown that the agents' budgets reach a stationary distribution after a certain number of time steps and present a power law distribution on the tail. Furthermore, the investment networks emerging from the model show that the networks present some of the typical characteristics of real-life networks like a high clustering coefficient and short path length. However, the degree distribution of the investment networks does not follow a power-law behavior which is usually found in real-world networks, but

rather a binomial distribution which is found in random networks. Although the main focus of this PhD thesis is more related to the area of computer science, the results presented here can be also applied to scenarios where the agent has to control other kinds of resources, such as energy, time consumption, expected life time, etc.

Zusammenfassung

Die folgende Arbeit befasst sich mit den Untersuchungen von Problemen der Optimierung von Ressourcen in Umgebungen mit unvorhersehbarem Verhalten, wo: (i) nicht alle Informationen verfügbar sind, und (ii) die Umgebung unbekannte zeitliche Veränderungen aufweist. Diese Dissertation ist folgendermaßen gegliedert:

Teil I stellt das Investitionsmodell vor. Es wird sowohl eine analytische als auch eine numerische Analyse der Dynamik dieses Modells für feste Investitionsstrategien in verschiedenen zufälligen Umgebungen vorgestellt. In diesem Investitionsmodell hängt die Dynamik des Budgets des Agenten $x(t)$ von der Zufälligkeit der exogenen Rendite $r(t)$ ab, wofür verschiedene Annahmen diskutiert wurden. Die Heavy-tailed Verteilung des Budgets wurde numerisch untersucht und mit theoretischen Vorhersagen verglichen. Darüber hinaus wurde gezeigt, dass der wahrscheinlichste Wert x_{mp} des Budgets einen konstanten Wert im Laufe der Zeit erreicht. Mit Hilfe von Simulationen wurde der Einfluss der stochastischen Faktoren auf den stationär wahrscheinlichsten Wert des Budgets untersucht. Die Ergebnisse der Simulationen deuten die Präsenz einer Skalierungsfunktion zwischen x_{mp} und den Parametern an, die die stochastische Dynamik charakterisieren. Die Ergebnisse der Simulationen wurden durch die Beschaffung einer analytischen Skalierungsfunktion bestätigt. Schließlich wurde die Entwicklung des Budgets des Agenten für reale Rendite aus Börsendaten für verschiedene feste Investitionsstategien und Einkommen untersucht.

In Teil II wurde ein Investitionsszenario mit stilisierten exogenen Renditen untersucht, das durch eine periodische Funktion mit verschiedenen Arten und Stärken von Rauschen charakterisiert ist. In diesem Szenario wurden unterschiedliche Strategien, Agenten-Verhalten und Agenten Fähigkeiten zur Vorhersage der zukünftigen $r(t)$ untersucht. Hier wurden Null-intelligenz-Agenten, die über technischen Analysen verfügen (wie z.B. Moving-Least-Squares), mit Agenten, die über genetischen Algorithmen verfügen, verglichen. Die Leistung eines Agenten wurde mit dem Wachstum seines Budgets nach einer bestimmten Anzahl von Zeitschritten gemessen. Um einen fairen Vergleich zwischen den Strategien zu garantieren, wurden die jeweiligen Parameter der Strategien an maximale Gewinne angepasst. Umfangreiche Ergebnisse von Computersimulationen wurden präsentiert, in denen nachgewiesen wurde, dass für exogene Renditen mit Periodizität: (i) das wagemutige das vorsichtige Verhalten überbietet, und (ii) die genetischen Algorithmen in der Lage sind, die optimalen Investitionsstrategien zu finden und deshalb die anderen Strategien überbieten. Diese Untersuchungen wurden erweitert, um die beste Investitionsstrategie für Rendite mit wechselnder Periodizität zu finden. Zu diesem Zweck wurde der genetische Algorithmus, durch die Variierung der Länge der Chromosomen und durch komplexere Gestaltung von Crossover- und Mutationsoperatoren erweitert. Durch dieses Verfahren kann der Algorithmus die korrekte Zuordnung der Anteile der Investitionen zu Mustern, die in Renditen vorkommen können, finden. Die Leistung dieser adaptiven Investitionsstrategie wurde mit der Leistung anderer Strategien, die als Referenz dienen, verglichen. Es wurde gezeigt, dass nach einer Reihe von Zeitschritten die adaptive Strategie eine Reihe von Investitionsstrategien finden kann, die einfache Strategien überbieten können, wie z.B. diejenigen, die immer einen konstanten Anteil investieren. Darüber hinaus wurde gezeigt, dass obwohl die adaptive Strategie keine Kenntnisse über die Dynamik der Renditen hat, sie trotzdem zu genauso großen Gewinnen führen kann, wie andere Strategien, die über diese Kenntnise verfügen. Schließlich wurde eine Erweiterung des Investitionsmodell für die Bildung von gemeinsamen Investitionsprojekten zwischen Agenten präsentiert. In diesem Szenario wurde jedes Projekt von einem Initiator angeführt, der versucht andere Agenten zu überzeugen in seinem Projekt zu investieren. Die Entscheidung des Agenten zu investieren hängt von den bisherigen Erfahrungen mit dem jeweiligen Initiator des Projektes ab. Der Einfluss der Parameter in der Dynamik des Budgets und in der Dy-

namik der Investitionsnetzwerke wurde untersucht. Es wurde gezeigt, dass das Budget der Agenten nach einiger Zeit eine stationäre Verteilung erreicht und eine Heavy-Tailed Verteilung des Budgets auftritt. Außerdem zeigen die Investitionsnetzwerke, die sich aus diesem Modell entwickeln, dass die Netzwerke einige typische Merkmale der realen Netzwerke zeigen, wie z.B. einen hohen Clustering-Koeffizienten und einen kleinen Durchmesser. Jedoch folgt die Verteilung von Knoten und die Anzahl von Verbindungen nicht jenem Potenzgesetz, das normalerweise in realen Netzwerken auftritt, sondern einer Binomialverteilung, die im Zusammenhang mit zufälligen Netzwerken steht.

Obwohl der Schwerpunkt dieser Dissertation im Zusammenhang mit dem Gebiet der Informatik präsentiert wurde, können die hier vorgestellten Ergebnisse auch in Szenarien angewendet werden, in denen der Agent anderere Arten von Ressourcen steuern muss, wie z.B. Energie, Zeitverbrauch, erwartete Lebensdauer, etc.

Contents

1. Introduction

This chapter gives the motivation for the research made during the PhD studies, a short review of the state-of-the art in machine learning and computational economics. Finally, the structure of the thesis is presented.

1.1. Motivation and Goals of the Thesis

For decades, researchers have dealt with the problem of optimization of resources in environments with unpredictable behavior.

In this PhD thesis, the main goal is to precisely investigate some of the problems related to optimization of resources in environments with unpredictable behavior where: (i) not all information is available and (ii) the environment presents unknown temporal changes.

In this PhD thesis, different types of environments are proposed in order to study the performance of different decision-making processes for optimization problems. The decision-making process is considered to be the strategy of an agent in its interaction with its environment. In terms of computer science, an agent is considered to be a software entity. For the purposes of this thesis agents may have different strategies and their performance may depend on: (i) available information and (ii) the processing capabilities (internal architecture) of the agent. Thus, the main goal in this PhD thesis is to investigate some of the problems related to the optimization of resources using software agents in environments with unpredictable behavior where: (i) not all information is available and (ii) the environment presents unknown temporal changes.

The essence of the problem is captured in Fig. 1.1 where $r(t)$ is the changing environment which influences $q(t)$ the strategy of an agent that has to adjust properly in order to reach optimal performance.

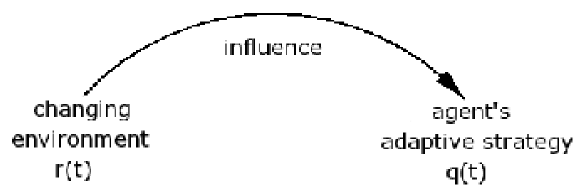


Figure 1.1.: The essence of the problem addressed in this PhD thesis. The agent, immersed in an environment with unpredictable behavior, $r(t)$, has to adjust its strategy, $q(t)$, to optimize its performance.

Note that for simplicity in this scenario, we do not consider the influence that the adaptive strategy has on the environment. Thus, the challenge for the agents is twofold: *first*, agents have to predict $r(t)$ as accurately as possible, and *second*, they have to adjust $q(t)$ to the proper values as quickly as possible. This is a complex and difficult task for environments with uncertain and fluctuating dynamics.

Thus, the investigations in this PhD thesis are mainly focused on studying the performance of different types of agents with different capabilities to adapt to their environment. In other words, we would like to know to what extent agents with more complex capabilities perform better than others with less complexity in different types of environments. In the following section, some contributions of computer science to this problem are described and the different types of agents considered in the investigations of this PhD thesis are presented. In Section 1.4 the testbed for strategies, performance and complexity is described. In Section 1.3 the testbed is related to some work done in the area of computational economics and econophysics. Finally, in Section 1.5 the outline of this PhD thesis is presented.

1.2. Contributions of Computer Science

For computer scientists, the problem of optimization of resources in environments with unpredictable behavior is usually an exciting topic which involves different steps. The first step is to model the system; for this we mainly use software entities. In order to avoid too much generalization or too much explicit description of the system, different types of software entities as well as parameters and properties are added step by step to the system until the latter has a similar desired behavior as the original environment. Afterwards, we generally handle the problem of finding an optimal solution by means of different types of algorithms and intensive computer simulations.

The main area of interest for this PhD thesis is the area of *artificial intelligence*. This research area focuses mainly on giving machines the ability to learn and understand their environment in order to solve problems and make decisions. One of the first contributions to this area is attributed to Turing [1950], who introduced the famous Turing test used to determine whether or not a machine may be referred to as intelligent. For a nice introduction to the area of artificial intelligence see [Negnevitsky, 2002]. Within this area, the investigations in this PhD thesis fall more specifically under the field of *machine learning*, which mainly investigates different learning processes that can be used in computer programs to improve their performance (see [Mitchell, 1997] for an introduction to this area).

There are different definitions for learning; I particularly like Gonzalez and Dankel [1993] definition:

> Learning is the improvement in the performance of a specific task (intellectual or physical) after previous exposure to that task or a related one.

On the other hand, *Machine Learning* is defined by Mitchell [1997] as follows: “Machine Learning is the study of computer algorithms that improve automatically through experience”.

Machine Learning approaches progressively learn from past experience; examples of these approaches are: *Genetic Algorithms* (GA), *Artificial Neural Networks* (NN), *Reinforcement Learning* (RL) and Expert Systems (ES).

Genetic Algorithms (GA) are powerful techniques inspired in natural selection that explore progressively from a large number of possible solutions which find after a number of generations, the best solution for the problem. Usually, a possible solution to the problem is represented by an artificial chromosome consisting of a number of genes, where each each can be binary (represented by 0 or 1), or a float-point number. In this manner, based on some defined evolution operators, a number of initial solutions are evolved iteratively (where each iteration is called a generation) and after a number of generations the best possible solutions to the problem are obtained [Forrest, 1996; Goldberg, 1989; Holland, 1975; Michalewiçz, 1999]. Other techniques that are also based on evolutionary computation are: evolutionary strategies and genetic programming. Evolutionary strategies, first introduced by Rechenberg [1973], were initially designed for solving parameter optimization problems using random changes in the parameters and are now also used in optimization problems [Ebeling, 1990; Schwefel, 1995]. On the other hand, genetic programming techniques do not evolve solutions that represent the problem but computer code that solves the problem [Koza, 1992]. Moreover, many of these approaches are applied to environments that are stationary; however, some researches have recently investigated the use of genetic algorithms in changing environments [Branke, 1999; Grefenstette, 1992; Harvey, 1992].

First introduced by McCulloch and Pitts [1943], *Neural Networks* (NN) are structures capable of processing information that resemble some of the processes of nervous systems. An artificial neural network usually consists of an input layer, an output layer and a number of intermediate layers where each layer consists of basic information-processing units called neurons. Each neuron is interconnected by weighted links with other neurons where and has a numeric value representing an activation energy. Given the inputs and numerical weights; a neuron computes an activation level and passes it as an output signal to another neuron. The way *NNs* learn the solution to a problem is via the iterative modification of the weights according to the desired input/output relationship. *NNs* have been attracting the interest of many researchers leading to different investigations regarding the dynamic of the neurons and their interconnections as well as their use; for example for pattern recognition and adaptive controllers [Carpenter and Grossberg, 1991; Hopfield, 1982; Kohonen, 1984; Minsky, 1954; Rojas, 1996].

On the other hand, *Reinforcement Learning* (RL) is a computational approach which deals with learning from interaction based on the idea that agents in an environment can improve their performance over time by processing the feedback they receive from the environment. This means that the agent accounts for its actions and rewards obtained during its interaction with the environment, where sometimes the rewards might be received with some delay after a sequence of actions [Sutton and Barto, 1998]. Most of the solutions to *RL* problems consist of implementing behavior rules to guide the agents in their decision making process. Usually, the following two distinct approaches are used to solve RL problems: (i) to search in value function space, or (ii) to search in policy space. Examples of these approaches are temporal difference methods and evolutionary algorithms. Temporal difference methods approach RL problems through the iterative update of reward estimation rules. For this purpose, several methods based on dynamic programming have been proposed and studied so far. However, they are designed for special cases when the problem can be formulated as a Markov Decision Process (MDP). The most popular among these methods is the Q-learning algorithm [Watkins, 1989]. On the other hand, evolutionary algorithms to solve RL problems are based on learning classifier systems. See Holland et al. [1986] for some of the foundations of this approach. More recently, Moriarty et al. [1999] investigated the application of evolutionary algorithms to RL problems, proposing different kinds of policy representations and problem-specific genetic operators. The authors show some of the advantages and disadvantages when using this approach to different RL problems.

Finally, *Expert Systems* (ES) consist mainly of a *knowledge base* a *database* and an *inference engine*. The knowledge base contains a set of rules of the type *if(condition) then(action)* and the database stores facts that are matched against the rules from the knowledge base by means of the inference engine [Newell and Simon, 1972]. This approach has been extended by many researchers where the main goal is to have expert systems that are able to manage uncertainty in a feasible way [Bonissone and Tong, 1985; Burkhard, 1998; Pearl, 1988].

Machine Learning has numerous applications, and in the following section some of the most important contributions related to signal processing, pattern recognition and forecasting are mentioned as they are relevant for the purposes of this thesis.

1.2.1. Signal Processing and Pattern Recognition

Evolutionary Computation

Other techniques developed from machine learning that are frequently used for investment decision problems are those based on *evolutionary computation* - for example, those using *genetic programming (GP)* and *genetic algorithms (GAs)* for portfolio management, inducing rules for bankruptcy prediction and assigning credit scoring, see [Bauer, 1994; Dawid, 1999]. For a recent review of the use of GAs for forecasting stock market prices and foreign exchange see also [Drake and Marks, 2002].

It has been shown that some investment strategies based on genetic programming techniques usually lead to profitable trading strategies. For instance, Pereira [1996] has used GAs for optimizing technical trading rules which are able to give a buy or a sell signal depending on the historical price and volume data. Neely et al. [1997] discussed the use of a GP approach for technical analysis in the foreign exchange market, reporting significant excess returns that could have been earned in currency markets using their GP approach. The authors also compare the performance of the trading rules found by the GP approach against standard statistical methods, reporting that the former detects patterns in the data that the latter is not able to detect. A similar approach is used by [Schulenburg and Ross, 1999, 2001], where the authors present a GP approach for modeling the behaviors of financial traders. The authors compare the performance of the GP with other well known investment strategies: buy-and-hold, trend-following, random walk and cash-hold. The authors present results for the stock of IBM, where the average wealth reached by the agents using the GP approach was in most cases higher than that of those using buy-and-hold and in all cases higher than that of those using the investment strategies of cash-hold, trend-following and random walk. A different approach is used by Kassicieh et al. [1998], the authors propose the use of a genetic algorithm to develop the closest to perfect foresight for guiding the switching between stocks and bonds. The authors use singular value decomposition (SVD) and neural networks to transform the initial data information and the output is introduced to a genetic algorithm which determines the best switching strategy. The authors report that non standardized SVD yields better results than neural networks. Moreover, Jiang and Szeto [2003] investigate the use of GP to calibrate the use of different technical analysis techniques based on moving averages in four different stocks from the NASDAQ. The authors assumed the rate of overall return to account for the performance of the strategies and compared these against benchmark methods such as random walk, buy-and-hold and exhaustive search. The authors report that in their experiments the GP approach is better than random walk and buy-and-hold, showing that the GP approach can result in good strategy sets. More recently, Schoreels and Garibaldi [2006a,b] investigated the use of a

GP to evolve agents' trading strategies on real historical equity market data using three approaches: technical analysis, the capital asset pricing model and a hybrid model of these two approaches. The authors report that the approach based on technical analysis performed better than the one based on the capital asset pricing model. However, the hybrid approach outperformed both non-hybrid approaches; supporting the use of multi-method based approaches in agent-based systems.

The previously mentioned approaches mainly use genetic programming (GP) approaches to optimize rules that are able to describe the environment and forecast the time series involved. However, they usually find strategies which are difficult to understand and which sometimes cannot be funded. Even though investment strategies that are based on genetic algorithms (GA) may be also difficult to abstract and to explain, I believe that they are more natural, understandable and flexible than those based on genetic programming techniques. For example, Szpiro [1997] proposes the use of GA to find equations that describe the behavior of a time series. The method permits global forecasts of chaotic time series using very little data. The author also discusses the fact that sometimes the equations found indicate the functional form of the dynamic that underlies the data. The algorithms are tested with clean and noisy chaotic data, as well as with the sunspot series. Alvarez et al. [2001] present DARWIN, an efficient evolutionary algorithm to approximate the functional relationship, in symbolic form, that describes the behavior of a time series. The authors report that DARWIN is particularly useful when the dynamic model that creates the time series is nonlinear, and discuss also the application of DARWIN as a predictor in a satellite-based ocean forecasting system. Kishtawal et al. [2003] use a GA for the prediction of summer rainfall over India. The authors report that the GA finds the equations that best describe the temporal variations of the seasonal rainfall, enabling the forecasting of future rainfall. More recently, Manimaran et al. [2006] used a GA in conjunction with discrete wavelets to forecast the trend in financial time series. The method proposed uses discrete wavelets to isolate the local, small-scale variations in the time series. Afterwards, the GA is used to find proper analytic equations predicting the time series. The authors report that the trends of the NASDAQ composite index and the Bombay stock exchange composite index are well captured by their approach.

Neural Networks

Several researchers have used artificial neural networks for forecasting time series, however, it has been noted that there are some disadvantages when using this approach. For instance, Zell [1995] refers to the following two disadvantages when using simple feed-forward neural networks to find patterns in time series: (i) the size of the input window is always fixed and (ii) for two sequences with the same information the network yields the same result independent of the context of the sequences in the whole time series. To circumvent these problems, some researchers have proposed different methods for dealing with the problem of a priori determination of the best topology for the network. One approach is to use a special class of neural networks, called ontogeny neural networks [Fiesler, 1994]. The main goal of this approach is to adapt the topology of the neural network automatically based on growing methods - in order to avoid getting trapped in local minima (adding new units) - and pruning methods - in order to improve generalization. A different approach is handled by Neuneier and Zimmermann [1998], in which the authors propose the use of sensitivity analysis to determine the best topology and the best way to train neural networks.

In economical contexts, neural networks have been used, for example, by Nikolopoulos and Fellrath [1994]. The authors propose a unified model that joins connectionist and logic programming paradigms for investment advising. By means of neural networks the authors detect the interest rate trends and use these as input to a deductive reasoning component which infers the most appropriate investment strategy depending on the investor's lifestyle, tolerance for risk and financial goals (short-, intermediate- and long-term goals). A survey of different approaches using neural networks for forecasting economical time series is given by Moody [1994, 1998]. Moreover, Magdon-Ismail et al. [2001] deal with the problem of how to learn the most important features from a large amount of data with noise. The authors use neural networks to find patterns from financial time series, where the main goal is to find changes in volatility. Lee et al. [2003] report the performance results of an auto-associative neural network for trend detection that was trained with the trend data obtained from the intra-day KOSPI 200 future price. As reported by the authors, simple investment strategies based on the detector achieved convincing gains. More recently, Castiglione [2004] presents a method based on a simple artificial neural network to forecast the sign of the price increment. The author reports a success rate of above 50 percent when predicting the sign of price increments for series from the S&P500, Nasdaq 100 and DowJones Ind. Kuo et al. [2004] provide another example. The authors propose the use of k-chart analysis and over-whelming self-organizing map neural networks and not only endeavor to improve the accuracy of uncovering trading signals but also to maximize the profits of trading.

Other examples of learning problems using neural networks for trading environments are investigated by Moody and Saffell [2001]. Some other authors have also focused their efforts on improving the performance of the neural networks in different investment instruments. For example, White and Racine [2001] included statistical resampling techniques to improve the performance of feed-forward neural networks. The authors report that the returns contain information that is can be used for prediction. However, the authors also show that the nature of the predictive relationships evolves over time. Prediction results for the foreign exchange market as well as for daily stocks from IBM are shown [White, 1998]. Another example is given by Zimmermann et al. [2001]. Here, the authors introduce a multi-agent approach for the modeling of multiple foreign exchange markets based on feed-forward neural networks. The novelty of this approach is the merging of economic theory of multi-agents with neural networks, which considers semantic specifications instead of being limited to ad hoc functional relationships. The authors report that their approach is superior to more conventional forecasting techniques when fitting data from the USD/DEM and YEN/DEM FX-Market. Some authors have considered some other techniques that are based on the theory of neural networks. For instance, Schittenkopf and Dorffner [2001] consider the concept of mixture density networks to investigate one of the central problems in finance which is to find better models for pricing and hedging financial derivatives. A well-known model, for example, for call and put options is the model attributed to Black and Scholes [1973]. Schittenkopf and Dorffner [2001] present a new semi-nonparametric approach to risk-neutral density extraction from option prices. The advantage of this approach is that it captures some stylized facts such as negative skewness and excess kurtosis. The authors show that this approach leads to significantly better results than when using the Black-Scholes model and a GARCH option pricing model, which includes a time-dependent volatility process. Van Gestel et al. [2001] use least squares support vector machine (LS-SVM) regression based on a Bayesian evidence framework in order to infer nonlinear models for predicting the time series and the volatility of different financial instruments. The authors report significant out of sample sign predictions with respect to the Pesaran-Timmerman test statistic when predicting the weekly 90-day T-bill rate and the daily DAX30 closing prices.

Reinforcement Learning

Recently, several models based on reinforcement learning have been used to explain experimental findings in strategic encounters (see [Camerer and Ho, 1999] for a review of them). On the other hand, some other researchers have proposed some learning theoretical foundations for evolutionary game theory, see [Börgers and Sarin, 1997, 2000]. More recently, Burgos [2002] analysed the relevance of adaptive learning in explaining phenomena like the fact that people tend to overvalue sure gains relative to outcomes which are merely probable and tend to accept bets when payoffs involve losses rather than gains. The author considers a type of adaptive learner first studied by Erev and Roth [1998]; Roth and Erev [1995] and shows, by means of simulations, that adaptive learning induces risk averse choices. More recently, Geibel and Wysotzki [2005] proposed the use of a risk-sensitive reinforcement learning algorithm to find the best policy for controlling under constraints and applied it to the control of a feed tank with stochastic inflows. Finally, some authors have shown that the performance of reinforcement learning algorithms can be improved by including supervised learning approaches [Uc-Cetina, 2007].

Methods Based on Data Mining

Other methods used to solve the problem of forecasting in time series include *Rough Set theory* and *Independent Component Analysis*. Rough Set Theory aims to discover and analyze data regularities and was originally proposed by Pawlak [1991]. For example, Skowron and Polkowski [1996] used rough set methods and Boolean reasoning techniques for deriving decision rules from experimental data, while Fernández-Baizán et al. [2000] used rough sets for short-term prediction of one variable, where inputs are historical values. The rules resulting from these approaches to predict time series are fairly similar to the kind of rules obtained using Genetic Programming approaches.

Other signal processing techniques that have been used for prediction in financial time series are independent component analysis (ICA) or blind source separation of multivariate financial time series such as a portfolio of stocks [Back and Weigend, 1997]. The key idea of ICA is to create a linear map of the observed multivariate time series in a new space of statistically independent components (ICs). Back and Weigend [1997] applied ICA to three years of daily returns of some Japanese stocks and compared the results with those obtained using principal component analysis. The authors report that the estimated ICs fall into the categories of infrequent large shocks and frequent smaller fluctuations. The authors also show that the overall stock price can be reconstructed effectively by means of a small number of thresholded weighted ICs.

Finally, note that methods based on the principle of entropy have been also proposed by many researchers, one measure that is commonly used is for example the measure of mutual information [Shannon and Weaver, 1949]. This measure is used for example for the analysis of symbolic sequences derived from the time series. The predictability of such methods has been investigated for different type of time series [Ebeling, 1993; Ortiz-Tánchez, 2004].

1.2.2. Testbeds for Machine Learning approaches

The previous sections presented approaches for signal processing based on machine learning algorithms which are used for specific problem-related scenarios. If other approaches are tried for the same problem, there may be difficulties in assuring a correct performance comparison between different approaches. For this performance comparison, different testbeds have been proposed in order to facilitate a comprehensible and reliable comparison of machine learning mechanisms and its complexity. Following Fromm [2004], complexity may have different definitions, however, for the purposes of this PhD thesis an entity like an agent or a learning mechanism is said to be complex if the manner they execute decisions or carry out tasks are difficult to analyse or understand. For example, the complexity of learning mechanisms has been investigated by Arthur [1994]. The author proposes the following three mechanisms for increasing complexity within an evolutionary approach: growth in evolutionary diversity (by coevolution), structural deepening, and capturing software. On the other hand, Dempster et al. [2001] compared the power of computational learning methods like reinforcement learning and genetic programming against simple heuristic-based methods for the problem of finding the most profitable trading rules in the foreign exchange market. The authors considered popular technical indicators as input for the strategies and showed that the performance of all methods resulted in significant profits when transaction costs are zero, whereas for non-zero transaction costs the genetic algorithm approach was superior. In another example, Gencay and Qi [2001] study the effectiveness of different techniques for daily pricing and hedging derivative securities from the S&P 500 index. The authors report that the use of Bayesian regularization produces less errors than when using the baseline neural network model and the Black-Scholes model. The authors also show that bagging provides the most accurate pricing and delta hedging, however, computationally, this the most demanding technique compared in this study. A more general testbed for machine learning algorithms is presented by Yannakakis et al. [2003]. The authors evaluate the performance, robustness and required computational effort of two different learning mechanisms in a multi-agent complex environment called “Flatland”. The authors show that for high performance values an evolutionary approach like genetic algorithms outperforms a gradient-based approach (like neural networks trained with standard online back-propagation algorithm) in robustness, performance and computational effort. The authors noted that neural networks had lower computational effort than the genetic algorithm for lower performance values only.

1.3. Contributions from Computational Economics

The problem of optimization of resources in environments with unpredictable behavior has been addressed not only by researchers from Computer Science, but also by researchers from other areas like economics and physics. Economists usually analyze the different processes that are involved in the system and propose different economic models to reproduce some of the properties observed in the environment. Once the system is modeled, economists usually find the maximum using different mathematical techniques, mainly based on calculus. On the other hand, physicists interests lay more in finding the most important features that generate the unpredictable behavior. Following the principle *keep it simple*, the system is modeled using basic characteristics and the rules for achieving optimal allocation of resources are found mainly by solving the equations: analytically, through simulations, or using randomized search methods like simulated annealing.

Thus, the approach taken in this PhD thesis may also draw the interest of at least two distinct research areas: *economics* and *physics*. Together these two yield the emerging research area of *econophysics*, which mainly makes use of methods from statistical physics to analyze data available from different economic sectors.

1.3.1. Econophysics

The area of *econophysics* studies economic systems from a physicist's perspective, providing interesting methods developed in statistical mechanics and theoretical physics to analyze financial markets, among others. Although, the main work of this PhD is in the area of computer science, the investigations have groundwork in concepts from econophysics like: power-law distributions, scaling and time series prediction. A nice introduction to some concepts in this field can be found in [Mantegna and Stanley, 2000; Schweitzer, 2003].

The motivation of this PhD thesis lies specially in the study of power law distributions in multiplicative stochastic processes.

Investigations Based on Statistical Physics

Several theoretical aspects of stochastic processes with multiplicative noise have been the focus of research in physics. For instance, Schenzle and Brand [1979] investigate different multiplicative stochastic processes from the perspective of statistical physics. The authors compare some general properties of multiplicative processes with those in additive ones. One important fact reported by the authors is that the most probable values of these processes have different behavior. For example, in the case of multiplicative fluctuations, the most probable values do not coincide with the deterministic stationary points. This means that the threshold conditions are not only determined by deterministic parameters but also by the strength of the fluctuations. More recently, Levy and Solomon [1996] investigated different multiplicative random processes leading to power laws distributions. For their investigations, they assumed a set of investors, whose wealth follows a simple multiplicative dynamic and applied a lower limit for the process values, which can be understood as subsidy so as not to allow individuals to fall bellow a certain poverty line. The authors present a master equation for the probability distribution of their wealth and obtain an analytical description of the asymptotic distribution of the process leading to a power law distribution. Their analytical results are validated using Monte Carlo simulations and experimental data. These investigations have been extended by other physicists using the Kesten process [Kesten, 1973], which extends the dynamics by introducing an *additive* stochastic term which is independent of the multiplicative coefficient. This extension has the advantage that the stochastic process is repelled from zero, provided some constraints on the additive term are satisfied, thus, avoiding the use of barriers or limits in the dynamics. For instance, Sornette and Cont [1997] study both multiplicative stochastic processes with a barrier and with an additive term, i.e. multiplicative processes repelled from zero. The authors describe the evolution of the process using a Fokker-Planck equation and propose a generalization of a broad class of multiplicative process with repulsion. Similar research for multiplicative stochastic processes with additive external noise has been conducted by Takayasu et al. [1997]. This research presents sufficient conditions to obtain steady power law fluctuations with divergent variance are presented as well as an equation to determine the exponent of the power law. The authors also prove the uniqueness and stability of the steady solution. Sornette [1998a] uses some of the previously mentioned results to extend some investigations done by Takayasu et al. [1997], where by means of characteristic functions the author studies the cases in which stochastic processes with multiplicative noise produce intermittency characterized by a power law probability density distribution. In a related study, Richmond [2001] find a steady state solution for a generalized Langevin equation of the form: $\dot{\varphi} = F(\varphi) + G(\varphi)\hat{\eta}$ where η is Gaussian white noise. These previous results are complemented by Malcai et al. [2002], in which a generalized Lotka-Volterra system under non-stationary conditions is studied. An analytical solution for the steady state is proposed and compared against numerical simulations for different parameter values. The authors conclude that the presence of a power-law distribution is a sign of "market efficiency", analogous to Boltzmann distributions in statistical mechanics systems. Despite their incredible simplicity, the dynamics of these processes have gained a considerable amount of attention in various fields. It was, for example, used in 1931 by Gibrat to describe the annual growth of companies – an idea extended in different works by economists [Sutton, 1997] and econophysicists [Aoyama et al., 2004]. In another example, Amaral et al. [1997] analyzed the *Compustat database*, which is comprised of publicly traded manufacturing companies in the U.S. between 1974 and 1993. The authors describe the distribution of growth rate of companies and propose a model for the growth of companies based on a multiplicative process. The distribution of wealth in a dynamic model of capital exchange is analyzed by Ispolatov et al. [1998], where amounts of money are exchanged between two individuals when they meet and different exchange rules result in different wealth distributions. These authors also discuss the wealth distributions in multiplicative processes and show that a steady state distribution is reached with random multiplicative exchange, whereas in a greedy exchange, where the rich get richer and the poor poorer, non-steady power law distributions arise. Marsili et al. [1998] study a simple model of dynamical redistribution of capital in a portfolio and review different results for multiplicative random walk processes. These include the typical and average values of the process for discrete and continuous time approaches. Also, the authors discuss the problem of multiplicative random walk in the presence of a lower wall introduced by means of an additive positive term. The authors show that for a particular lower wall term, the analytic solution for the Fokker-Planck equation that describes the process leads to a Boltzmann distribution. The authors demonstrate that the strategy they propose to dynamically redistribute the capital results in a larger typical growth rate than a static "buy-and-hold" strategy, where the capital is initially equally distributed among the assets with no redistribution. Another example is presented by Carlson and Doyle [1999]. The authors introduced a mechanism for generating power law distributions based on natural selection. The emergence of power laws are due to tradeoffs between yield, cost of resources and tolerance to risks. The authors also considered lattice models where percolation problems and sand piles were investigated. Moreover, Dragulescu and Yakovenko [2001] analysed the income distribution of individuals in the the United Kingdom and the United States. The authors found that the majority of the population is described by an exponential distribution, whereas the wealth of the minority, corresponding to the richest people, follows a power law. A different approach is to consider models for capital exchange where buying and selling of goods between individuals is simulated with agents. For example, Iglesias et al. [2004] investigated different capital exchange models and try to explain the emergence of power-law wealth distributions. The authors focused their investigations on the influence of the risk aversion of the agents in the wealth distribution. The authors used the term "risk aversion" to describe the fraction of capital that the agent saves, where the rest is exchanged with other agents. Note that in the language of economics, this fraction of resources saved is referred to as a measure of the agent risk aversion, see [Chakraborti and Charkrabarti, 2000; Gusman et al., 2005; Scafetta et al., 2004a]. Both models investigated by Iglesias et al. [2004] considered that the poorer of the two partners is favored in each transaction with a given probability. This assumption is made in accordance with the notion of stable society presented by Scafetta et al. [2004b]:

> In order for a society to be stable rather than the poor being exploited in trades with the rich they must have an advantage, at least in a statistical sense.

Iglesias et al. [2004] report a correlation between wealth and risk aversion showing that a more equitable society emerges in the case in which extremal (minimum) dynamics are considered, i.e. one of the partners is an agent with minimum wealth and the other is chosen at random. On the other hand, when Monte Carlo dynamics are considered, i.e. both agents in a transaction are chosen at random, the authors report that a capitalist society emerges. [Gusman et al., 2005] extend these investigations, focusing on the characterization of the most probable budget of the poorest agent in the population. To do this, the authors use the Gini coefficient, which is a measure of statistical dispersion used in this study to measure the inequality of income distribution. The authors also consider connectivity between the agents and find that if the average connectivity is increased then the power-law distributions emerge in most of the different cases they considered. In a related study, Scafetta et al. [2004a] investigates a model with both investment and trade mechanisms which favor the less wealthy agent, reproducing the stratification of the society into poor, middle and rich classes respectively. Moreover, Fuentes et al. [2006] study a model for wealth distribution where the partners in a transaction have some previous knowledge about the return that may be received and adjust their risk aversion accordingly. The authors report that in their model, when agents engage in a rational behavior, i.e. risking proportionally to their probability of winning, then a relatively equalitarian society emerges. Unfortunately, this approach does not agree with empirical data which suggests the contrary, i.e. individuals with an irrational behavior, risking more than reasonable. More recently, Moukarzel et al. [2007] presented some analytical investigations for these type of models. The authors derived the condensation conditions and compared them against the numerical results. For the non-condensed phase, the authors show that the equilibrium wealth distribution corresponds to a power law, in which exponents are also derived analytically. Finally, the authors discuss that the effect “rich get richer” is probably due to the multiplicative dynamics in the model.

Different applications for the multiplicative stochastic processes have been also considered. For example, Blank and Solomon [2000] show the presence of scaling effects in different systems like cities population, finance markets and Internet sites. More specifically, Huang and Solomon [2002] have modeled financial markets as a stochastic multiplicative system composed of finite asynchronous elements and shown that the wealth fluctuations of the system can be described by truncated Lévy-like distributions. Moreover, the authors also show that the origin of other properties of the returns in financial markets, like power law distribution and long-range persistence of volatility correlation, are caused by the cross-correlation between relative updated wealths. Another interesting investigation for characteristics of financial time series was conducted by Gorski et al. [2002]. The authors analyzed data from the Deutsche Aktienindex (DAX) and find different power law behavior governing the distribution of the returns with exponents exceeding that of Lévy regime. On the other hand, Mizuno et al. [2002] analyzed databases of Japanese companies' incomes, confirming Zipf's law in the income distribution, and found that small and large companies have similar statistical chances of growth. This has been shown using the discrete version of the linear Langevin equation for a stochastic variable, see [Takayasu et al., 1997], to estimate the exponents of the power law distribution using data of growth rates. In a related study, Sato and Takayasu [2002b] constructed an analog electrical circuit that generates fluctuations for which the probability density function shows a power law in the tail. The circuit is constructed based on the theory of random multiplicative process and the authors show that the fluctuations have statistical properties similar to those from the foreign exchange rates. Multiplicative models for firm dynamics from the physics point of view are also treated by Richiardi [2004]. Using simulations, the author investigates the modifications to the standard multiplicative model of firm dynamics needed in order to obtain stable distributions of firm size. To this end, the author shows that to this end either heteroskedasticity in the growth rates or entry/exit mechanisms have to be assumed. A related investment strategy for improving firm growth is presented by Takayuki et al. [2004], where both real data and numerical data from a multiplicative stochastic process process are used to estimate the average growth of a company's income. Different investment strategies are evaluated in the numerical model leading to a strategy that numerically gives the best average annual growth.

Investigations Based on Random Processes

The importance of multiplicative stochastic processes was also reflected by several mathematical investigations. For instance, some fundamentals of the theory of multiplicative schemes (M-schemes) based on the multiplication of random variables were presented by Zolotarev [1962]. In his paper, the author describes a class of M-infinitely divisible distribution laws and general limiting theorems, which are proved using characteristic transformation of the random variables. And the convergence of such multiplicative schemes to log-normal distributions is shown by Bakshtis [1972].

As it was mentioned earlier, Kesten [1973] studied the limit distribution of the solution of a difference equation with multiplicative and additive coefficients represented by random matrices and vectors respectively. Some mathematical properties of the multiplicative stochastic processes with an additive term were also studied by Vervaat [1979]. The author presents some mathematical proofs for the converge of the process to a power law distribution. The author makes also reference to some applications of the processes modeled by a simple multiplicative dynamic with an additive term. In general, the main variable represents a stock of objects, the additive term represents a number of objects added to the stock and the multiplicative term represents the decay or increase of the stock at every time step. For example, Lassner [1974] assumes that the process represents the value of a savings account and the additive and multiplicative term represent the deposit made and the interest factor at every time step respectively, which may fluctuate stochastically with time. In a related study, Perrakis and Henin [1974] evaluate risky investments where the multiplicative terms are the cash returns with random timing. Another example is the study conducted by Uppuluri et al. [1967], in which the authors assume that the process represents a stock of radioactive material, the additive term represents the quantity added or taken away at every time step and the multiplicative term represents the natural decay of radioactivity. This latter quantity depends on the material that is added or removed at every time step, resulting in a model with multiplicative and additive stochastic coefficients.

The previously mentioned investigations are extended by Brandt [1986]. In this study, the stability of the process with multiplicative and additive terms is investigated in the case of stationary coefficients. Haan and Karandikar [1989] deal with the problem of transforming multiplicative stochastic processes with an additive term from discrete to continuous time, i.e. the authors transform the difference equation of the model into a differential equation. The authors also examine the cases for which the continuous-time process converges in distribution. This previous investigations are complemented by Haan et al. [1989]. Here, the authors study multiplicative additive random process as a generalization of a first order

ARCH process. The authors also analyze the extremal behavior of the process, analytically determine the existence of an extremal value limit law and numerically compute the extremal index of the process. Redner [1990] present a tutorial for random multiplicative processes, in which the author shows that the logarithm of the probability of the process follows a Gaussian distribution, i.e. the distribution of the product has a log-normal distribution. The author emphasizes the differences in asymptotic behavior between a random product of variables and a sum of random variables. A more practical approach is taken by Maslov and Zhang [1998], in which the authors try to determine the portfolio strategy that provides the maximal typical long-term growth rate of investor's capital. In their approach, the evolution of the capital of an investor is modeled by means of a multiplicative process, in which the stochastic fluctuations are due to risky assets modeled by multiplicative Brownian motion processes. In a related study, Kotlyar and Antonov [2000] discuss the use of multiplicative schemes based on multiplicative random variables for financial calculations. These include determining inflation rate, market value of securities and investments, variability of production indices and depositary accumulations. The authors are able to construct confidence intervals for estimates of geometrical expectation of logarithmically normal distribution for known and unknown standard relative deviation. In more recent contributions, Horst [2001] investigated the stability of linear stochastic difference equations with multiplicative and additive terms for non-stationary coefficients. This previous work is extended for strategically controlled random environments [Horst, 2004] and complemented by the investigations done by Saporta et al. [2004], in which the authors study the behavior at infinity of the tail of the stationary distribution of the process with random coefficients. The authors present an extended class of multiplicative coefficients that satisfy irreducibility and proximality, which lead to a heavy tail behavior of the process.

1.3.2. Computational Intelligence in Investment Strategies

In the previous section we presented some state-of-the art techniques regarding machine learning and some of the applications and test-beds that are more related to the area of robotics, control and automatization were mentioned. In this section some contributions made to the area of agent-based computational economics are described, in which different machine learning algorithms are also frequently used for modeling behavior or cognitive processes for artificial agents. We start by presenting some general work that has been done in the areas of decision making and utility theory and afterwards some research work done related to the simulation of agent intelligence for artificial financial markets is presented.

Decision Making and Utility Theory

When we talk about decision making with uncertainty and the use of utility theory to measure the preferences of an individual when choosing between different prospects, we need to start with the concept of risk. The term *risk*, according to many researchers, can be defined as a measure of uncertainty and is associated with the statistical concept of *variance* (see [Holton, 2004; Knight, 1921; Rothschild and Stiglitz, 1970] for more details about the definition of risk). Usually, in the fields of decision making and utility theory under uncertainty, one can see two research branches. In the first, researchers focus their attention on the important task of how to measure risk, leading to different types of measures [Artzner et al., 1999; Pratt, 1964]. In the second one, researchers focus their attention on finding appropriate strategies when dealing with risky processes [Kahneman and Tversky, 1979; Kahnemann and Riepe, 1998; Kelly, 1956; Maslov and Zhang, 1998; Tobin, 1958]. In the following, both approaches are reviewed more in detail.

Measures of risk aversion:

Some researchers have focused their investigations on trying to find different measures of risk aversion. For instance, Pratt [1964] discussed the concept of risk-aversion, where a decision maker has to choose between receiving a random or non-random amount. According to the author, risk aversion is present when an agent prefers to receive the expected value of a lottery rather than to participate in it. In this article, the author analyses two cases of risk aversion: local and proportional risk aversion. For each case, Pratt proposes different measures of risk aversion, which are known as ARA (absolute risk aversion) and RRA (relative risk aversion). The author also presents equations to calculate the risk premium and the probability premium based on these measures. Concepts like constant risk aversion as well as increasing and decreasing risk aversion are also discussed for both cases. A similar study with extended examples and detailed explanations is presented by Arrow [1965] and some economic concepts of risk are treated by Rothschild and Stiglitz [1970]. More recently, Montesano [1991] considered Arrow-Pratt's risk aversion measure and analyzed this for Expected and Non-expected Utility, noting that measuring risk aversion through the ratio between the risk premium and the standard deviation of the lottery captures the main feature of risk aversion. Moreover, Artzner et al. [1999] presented four desirable properties for measures of risk and demonstrated the universality of scenario-based methods for providing coherent measures. The authors also discuss the use of previously proposed measures of risk aversion and classify investors by their behavior towards risk (e.g. risk-averse, risk-neutral or risk-seeking behaviors). Finally, [Holton, 2004] presents a detailed review of the definitions of risk and the subjective vs. objective interpretations of probability along with some historical remarks.

Strategies for risky processes:

As previously mentioned, some other researchers focus their attention on finding not only appropriate measures for risk aversion but also appropriate strategies to control *risk-exposure* in environments with uncertainty. It is important to note that the investigations in this thesis are more in line with this approach. Finding a proper investment strategy in environments that are uncertain and where fluctuations are present is a complex and difficult task and methods from artificial intelligence are often utilized for this task. For example, choosing to avoid investment may lead to the loss of major opportunities to win large amounts of money. On the other hand, choosing to invest large amounts of money may lead to situations where the chances of losing the entire investment are very high. Thus, the task of finding an appropriate strategy that balances these two extrema is by far not trivial. The usual approach to this problem is to find a proper investment policy based on the average return and the volatility of the asset. For instance, a first description of optimal strategies for placing bets in gambling is presented in the seminal contribution of Kelly [1956]. Take, for example a game, in which the odds are in the favor of the gambler but with a large uncertainty factor. The author shows that for a gambler knowing the result of the bet with a given probability, if the gambler speculates the result of the bets in advance, he can maximize the exponential rate of his wealth by betting a fraction of his capital. The author also shows that in such scenarios the profits of the gambler will be larger if he bets a fraction of his budget rather than if he bets his whole capital at every time step. In modern finance, investing in risky assets is similar to gambling as shown by Breiman [1960], who described multi-asset optimal investment strategies for risky assets.

A different approach for the problem of portfolio optimization is the one addressed by Sornette [1998b], who uses tools developed in statistical physics to address the problems of risk controlling and optimal diversification in finance and insurance. The author demonstrates for different scenarios that in order to perform a complete assesment of risk management, the agent needs to consider the full distribution of price variations, not only the conventional mean-variance approach. The author also discusses the role of large deviations in multiplicative processes and presents different optimal strategies that consider additional parameters like the time-horizon and the aversion to rare and relatively large risks. A similar approach is taken by Crama and Schyns [2003], in which the authors consider a simulated annealing meta-heuristic approach to find the optimal diversification strategy for a portfolio selection model based on a classical mean-variance portfolio model and enriched with realistic constraints like lower and upper limits on held proportion of the asset, lower limits on the variations of the holdings (representing investors not willing to modify the portfolio for only a few assets) and maximum number of assets for easy and faster management. In a related study, Urbanowicz and Holyst [2004] use properties of coarse-grained entropy to analyze the noise level for the Dow Jones index and some stocks from the New York Stock Exchange, showing that an investment strategy based on a threshold for minimal noise results in average positive returns.

Liu et al. [2003] takes a different approach combining historical data and finance theory. Here, the authors consider soft data which includes news, announcements from the government or industry and political events to feed a set of linear belief functions for portfolio evaluation. The authors report outperformance in comparison with some previous similar approaches that include not only historical data but financial knowledge as well.

More recently, some researchers considered assets with prices modeled by means of a multiplicative random walk process and described an investment strategy to readjust the portfolio [Merton, 1990]. This has been extended by Maslov and Zhang [1998] for a general distribution of return per capital. All these contributions consider exogenous returns, which are drawn from a probability distribution, or modeled by stochastic processes. These approaches are extended by Kahnemann and Riepe [1998], who investigate different utility functions for preferences and beliefs for investment advisors and by Thorp [2000] who analyses the Kelly Criterion for simple gambling games like blackjack and more complex scenarios like the stock market. Another example of risk-controlling under uncertainty is presented by Rogers [2001]. In this study, the author models the effect of infrequent policy review for an agent that re-balances its portfolio and consumption behavior only at given periods of time. Different investment strategies under uncertainty are also analysed by Thijssen [2003]. The author presents the effect of information streams on investment decisions and the role of information spillovers in strategic investment. The author also investigates the role that bounded rationality has on the evolution of different type of markets.

Another interesting problem is that of learning and expectation formation in macroeconomics, in which the expectations of the risky processes influence the time path of the economy and on the other hand the time path of the economy influences the expectations. In order to analyse this problem a rational expectation approach is usually assumed which balances both relationships [Evans and Honkapohja, 2001]. Moreover, some researchers have used computational intelligence approaches to investigate the convergence of models and learning approaches to a Rational Expectation Equilibria *(REE)*. One important contribution is due to Arifovic [1994, 1996]. Here, the author demonstrates that the genetic algorithms can be effectively used in models to converge to the REE.

Testbed scenarios:

The typical scenario used to study decision-making processes with uncertainty and, more specifically, those related to investment strategies, is to allow an agent to choose between investing in a risk-free asset or in a risky asset, see [Pratt, 1964; Tobin, 1958]. For example, Tobin [1958] showed that sometimes it may be more reasonable to invest in a risk-free asset as a means to transfer wealth over time. This was investigated more recently by Loomes [1998], who analyzed some fundamental assumptions about rational decision making in general economic scenarios. More specifically, Potters and Bouchaud [2005] have shown that it may be better not to follow trends as this strategy sometimes leads to more losses than gains. On the other hand, assuming a model with no consumption, Hens and Schenk-Hoppé [2006] have shown that those agents investing in risk-free assets will be driven out of the market in the long run by agents investing in risky assets.

Other researchers have investigated the performance of different strategies following a different approach, for example based on two types of agents: agents which only react on external changes (also known as "zero-intelligence agents" [Farmer et al., 2005]) and, agents which have a complex internal architecture [Gode and Sunder, 1993]. For instance, Lohmann and Baksh [1994] analyzed the performance of different strategies for dealing with risk in different scenarios. The authors refer to the problem when analyzing the relative performance of different decision procedures in attaining the decision maker's financial objectives. From their point of view, it is not clear how to achieve this while also dealing effectively with risk in long sequences of capital rationing decisions. The authors performed empirical investigations by conducting Monte Carlo computer simulation of long sequences of capital rationing decisions and used this data to evaluate the performance of six different capital budgeting decision procedures for dealing with risk. The performance was measured in terms of the relative effects on the capital growth rate and risk of ruin. Another interesting approach was presented by MacLean et al. [2003], who compared a variable planning horizon approach with the standard Value-at-Risk methodology with a fixed time horizon. The authors show that their proposed strategy has greater expected return with equivalent downside risk which may be attributed to the fact that this strategy comes into action when the wealth deviates from the expectations. Many textbooks also address the topics of risk aversion, choice under uncertainty and adjustments on risk (see for example [Nicholson, 1992; Pindyck and Rubinfeld, 1992]).

A different testbed is proposed by Helmbold et al. [1996]. Here, the authors present an on-line algorithm based on a multiplicative update rule that performs as well as the best constant-re-balanced portfolio when the actual market outcomes are known. The authors performed experiments with real returns from the New York Stock Exchange and showed that their strategy performed better than the best single stock and the universal portfolio selection algorithm proposed by Cover [1991]. However, the authors do not consider trading costs and they also address the problem that most of the investigations done on investment strategies assume that the market is stationary. This is an unrealistic situation as markets are changing constantly and comparing strategies against the best single constant re-balanced portfolio does not ensure the best performance of a strategy for all types of markets and for all time steps. In a similar study performed by Dempster et al. [2003], the authors evaluate the performance of dynamic investment strategies based on fixed-mix portfolio rules. The authors test their approach for stationary stochastic processes and show that for stationary markets their strategy yields exponential growth with a probability of one. In a related study, Gaivoronski and Stella [2003] propose a family of adaptive portfolio selection policies to re-balance the portfolio when the number of decision periods is large and new information about market arrives during each period. The strategy re-balances the current portfolio by adopting the portfolio from a family with best performance on the past data. If no transaction costs are present, the performance is the same as that of those strategies with full knowledge of the future. On the other hand, for nonzero transaction costs, the performance is asymptotically congruent to the performance of the strategy with perfect knowledge of the future.

Parkes and Huberman [2001] propose a different testbed for optimal portfolio diversification, in which three different portfolio selection strategies are compared. The first one consists of a constant re-balanced portfolio which maintains the same proportion of wealth invested in each stock. The second is an adaptive strategy in which an agent's portfolio is updated based on its recent performance and stock changes by means of a learning rate parameter. Finally, the third strategy performs a multi-agent cooperative search, adding communication between agents that promotes cooperation through hint exchange. Experimental results show better performance for strategies that include communication in simulated stock market time series based on geometric Brownian motion. When using the Capital Asset Pricing Model to generate time series, communication between agents seems to be unnecessary to outperform the non-communicative strategies.

Another interesting topic in the area of computational investment strategies is the identification of strategies acting in the market. For example, Popkov and Berg [2002] propose a method of identifying an economic agent's competitive strategies based on empirical data gathered from the Russian financial market.

Artificial Financial Markets

When referring to the simulation of artificial markets, one of the most referenced investigations in this area is the study done by Bachelier [1900], who in his PhD thesis proposed the use of Brownian Motion for modeling stock market time series. However, when referring to machine learning and artificial markets, we usually think of a software agent buying and selling artificial stocks. Some researchers have been interested in the properties that emerge in an artificial market from the interaction between agents, however other researchers have focused their efforts in finding the investment strategies that result in larger profits in such environments. In this section, we present some work done in both areas, however, for the purpose of this PhD thesis we start by presenting some investigations related to the problem of finding optimal strategies for profitable economic trading, and address research work done in analyzing the stylized properties that emerge from artificial markets at the end of the section.

Some researchers have been interested in defining intelligent trading algorithms and optimal pricing strategies. For example, Gode and Sunder [1993] reported one of the first experiments on a continuous double-auction system with computational agents. A continuous double auction is an auction for standardized units in which the offers to buy and sell units are posted and continuously matched by a market maker. A broader approach was taken by Rust et al. [1994] who made a comparative analysis of thirty trading algorithms which were part of a double-auction tournament held at the Santa Fe Institute between 1990 and 1991. The winner in this contest was one algorithm where the strategy was to wait while others negotiate, then when the bid comes, jump into the game, steal the deal and ask prices sufficiently close to the bid. Another approach for trading study is to consider different kind of traders, some of whom have access to more information than others. Arthur [1999] shows that speculations and herding behavior can also be observed when computers negotiate with each other in a fully autonomous way. The author investigates artificial programs that act like investors, making bids and offers and generating and discarding expectations hypotheses about the price of artificial stocks. The author also presents the “El Faron Bar” problem, relating this to financial markets and the strategy that in the beginning one should perform short steps while trying to learn more about the complex environment. Howitt and Clower [2000] studied the role of a trade specialist in a model of a decentralized market. They define a trade specialist as a trader who can reduce the costs of searching, bargaining and exchange by setting up a trading facility communication medium between non-specialist traders. They determined that in just over 90 percent of their simulations, a "fully developed" market economy emerges. Ingber and Mondescu [2001] described a process for the optimization of trading in physics-based market models. The authors developed stochastic nonlinear dynamic models for futures trading systems and used recursive and adaptive optimization with adaptive simulated annealing in order to fit the parameters of the trading models. Nagel et al. [2004] investigate a simple model of an economy, in which each agent produces exactly one good and is able to buy many other goods. The main goal of the agents is to find the correct balance between work and consumption. The authors present analytical solutions and simulation results for their model. Particularly, the authors find that a well-defined market only emerges when prices adapt on a slower time scale than consumption noting the importance of timescales for economic markets.

Some of the previously mentioned artificial market models have been used to investigate the performance of investment strategies based on adaptive or evolutionary approaches like reinforcement learning algorithms, neuronal networks, genetic algorithms and genetic programming, see Section 1.2.1. The main goal is to represent learning processes of computational agents in which different local learning schemes are placed in each individual agent or in a group of agents that evolve their strategies based on their own local benefits [Andreoni and Miller, 1995; Arthur et al., 1997; LeBaron, 2000; Takahashi and Terano, 2003]. For more applications of evolutionary learning in studying the behavior of agents in a broad array of social settings, see [Chamley, 2003; Gilbert and Troitzsch, 1999a; Gintis, 2000].

On the other hand, some researchers have focused their attention on the regularities found in artificial markets and the properties of the returns generated by means of constant trading between heterogeneous agents. Some of these approaches agree that the best way to find a good investment strategy is to take some empirical features of real financial markets into consideration. These include fat-tailed asset return distributions, high trading volumes, persistence and clustering in asset return volatility, cross correlations between asset returns, trading volume and volatility, among others. Some related research has focused on the problem of finding the relationship between the properties and mechanisms included in the agent’s investment strategies at the micro-level and the emerging empirical features of the artificial financial markets at the macro-level. For some interesting surveys on the earliest ACE (Agent-based Computational Economics) financial market studies see [LeBaron, 2000; Levy et al., 2000]. Moreover, Bak et al. [1999] analyze the dynamics of price determination in a simple toy model where agents try to estimate the quantity of goods that their neighbor will order and at which price. The authors also analyze the ways in which agents determine the value of money based on maximization of utility functions. Another interesting contribution is due to Blok [2000], the author presents different simple models of stock exchange and analyse them analytically and by means of simulations. The author shows that the decentralized model reproduce different properties that can be observed empirically in real markets. Moreover, it is shown that these empirical phenomena emerge endogenously from the interactions between agents and not from a stochastic driving force. This suggests that the complexity of the market dynamics does not arise from the complexity of the agents itself but from their interactions, as it is shown that some of the properties (like fat taills and long-range correlations in volatility) emerge even from the interaction between simple agents.

Some other researchers have shown that in different artificial financial markets, chartist behavior is outperformed by fundamentalist behavior and that the presence of chartist behavior may be the cause for some stylized characteristics of exchange markets [Anufriev, 2005; Day and Huang, 1990; Farmer and Joshi, 2002; Follmer et al., 2005; Potters and Bouchaud, 2005]. However, not only chartists and fundamentalists are of interest for the research community. For instance, LeBaron [2001] uses a genetic algorithm to co-evolve the collection of trading rules available to the agents, showing that this model is able to generate financial features from real markets. The author proposes to fit the parameters of an artificial financial market to match empirically observed regularities for real financial markets. This is done by calibrating an agent-based computational stock market model, which is used to aggregate macroeconomic and financial data. In this model, the artificial investors have different memory lengths which they use to evaluate the past performance of their trading rules. Amir et al. [2002] also analyzed market selection and the survival of different investment strategies in a toy model for a financial market. A similar approach is the one taken by Alfarano et al. [2003], who show that the universality of stylised characteristics of financial markets lies in the strategies. They found, for example, that by connecting some micro variables of the market to a macro variable, in speculative markets, the tail of the distribution becomes fatter if the importance of the herding mechanism increases. In a related study, Takahashi and Terano [2003] used an agent-based approach to analyze how investors' interactions and investment strategies based on behavioral finance affect asset prices. The authors compare fundamentalist and non-fundamentalist investors in a virtual financial market, showing that fundamentalists outperform non-fundamentalists in most of their experiments. Additionally, Lawrenz and Westerhoff [2003] present a Model where market participants apply technical trading rules or fundamental trading rules according to a weighting scheme. Strategies are selected based on a genetic algorithm. The results exhibit features typically observed in the foreign exchange market. More recently, Lux and Schornstein [2005] investigated the use of genetic algorithms for updating agents' decision rules in a model of exchange rate formation. The authors show that for some particular parameterizations, the dynamics of the model are similar to those exhibit by empirical data.

Other related contributions investigate, for example, the impact of shocks in banks [Drehmann, 2005]. In this study, different scenarios for stress testing in credit exposures are simulated and analyzed. The author's main interest is the impact of systematic risk factors in driving correlated losses. His study shows that the UK banking system is robust and that even under the worst macroeconomic conditions, the expected losses of banks are not high enough to cause a bank failure. The main problematic of the micro-macro link is also discussed by Schillo et al. [2000], where the authors describe some micro-macro relations in Distributed Artificial Intelligence (DAI) and sociology. In addition, LeCorre and Mischke [2005] detail the process of innovation management in three layers: the microscopic, mesoscopic and macroscopic layers. The authors present the particular problems for each layer and discuss how to solve them using mainly strategies that control the probability of an innovation success by computing and optimizing the respective success-changes.

Formation of Trade Networks

In recent years, the topological structure and the evolution of complex networks have been of great interest for many researchers [Albert and Barabási, 2002; Erdös and Rényi, 1959; Milgram, 1967; Newman, 2003; Xulvi-Brunet, 2006]. Some other researchers have focused their investigations on the analysis of different properties of networks that emerge from the interaction of agents [Axelrod, 1997; Bornholdt and Schuster, 2002; Dorogovtsev and Mendes, 2003; Ebel et al., 2003; Wasserman and Faust, 1994; Watts and Strogatz, 1998]. Some of these researchers have focused on the endogenous formation of trade networks [Albin and Foley, 1992; McFadzean et al., 2001; Tesfatsion, 1997; Vriend, 1995]. A key concern in these studies is the emergence of a trade network among a collection of buyers and sellers, who adaptively select their trade partners. In this approach, buyers and sellers make this selection by assesing at their experiences with previous partners. For example, Kirman and Vriend [2001] shows a model of the wholesale fish market in Marseilles. The authors investigate the price dispersion and widespread buyer loyalty to sellers by analyzing repeated business. They use a version of Holland's classifier system [Holland, 1992] to separate decisions for each agent. The authors report that price dispersion and loyalty emerge as a result of the co-evolution of buyer and seller decision rules. The authors also have seen that in their model, buyers learn to be loyal as sellers learn to offer a higher payoff to loyal buyers.

Other researchers have focused their research on trying different models in which the trading between agents result mainly in small-world type networks. For instance, Wilhite [2001] used a model of a bilateral exchange economy and found that there are micro-level incentives for the formation of small-world trade networks. A different approach taken by some researchers is to fix the topology of the network by concentrating the investigations on analyzing the performance of the market. For example, da Silva et al. [2005] present a simple artificial financial market where the investment of the agent depends on its own motivation level and the motivation level of its neighbors. The authors assume a ring interaction network for the agents and characterize the performance of the agents in the market using the concept of persistence [Majumdar et al., 1996], which measures the chance of keeping a positive balance relative to the initial amount invested. In their model, returns depend on both a random variable and on the investment. The authors study the performance of the agents for cases in which a fraction of the agents are conservatives or deceivers, finding that if the fraction of conservatives is increased, a phase transition in the persistence occurs and that a small ratio of deceivers makes the market profitable. Finally, the authors report that if the motivation is updated depending on the returns, agents become more risk-averse and the market becomes more persistent, but the profits and the losses become smaller.

Another interesting approach that applies behavioral models for agents is the area of *Agent-based Computational Economic* (ACE) for labor markets (see [Tesfatsion, 2001b, 2002] for a review). For example, different labor market models have been proposed to study the relationship between market structure and worker-employer interaction networks [Tesfatsion, 1997, 1998, 2001a]. In these investigations, workers and employers repeatedly search for preferred work-site partners in which interactions are modeled by prisoners' dilemma games. Moreover, the agents evolve their search strategies over the course of time based on their earnings in past interactions. Tesfatsion uses descriptive statistics to study the correlations between market structure and worker-employer network formations. Also of interest is the analysis of job concentration (number of workers to number of employers) and job capacity (total potential job openings to total potential work offers). Tesfatsion finds that if the job capacity is fixed, the changes in job concentration have only small, unsystematic effects on relative market power levels. However, the distribution of the networks exhibits two or three sharp isolated peaks corresponding to distinct types of worker-employer interaction networks. This means that the interaction effects are strong, which, Tesfatsion comments, may help to explain the "excess of heterogeneity" observed in labor markets.

Other types of networks that are of interest for many researchers are those between buyers and suppliers. For instance, Klos and Nooteboom [2001] investigate the evolution of interaction networks among buyers and supplier firms that repeatedly choose and refuse their partners on the basis of continually updated anticipations of future returns. In this model, firms have to decide between searching for suppliers for the production or handling production themselves. The authors show that if there is more product differentiation, firms tend to choose to produce themselves due to higher switching costs and scale effects. Rouchier et al. [2001] investigated a more complex model dealing with the conditions that determine how nomadic herdsmen access pasture lands. A key finding is that the grazing patterns and interaction networks established among herdsmen, village leaders, and village farmers tend to be very regular. The authors also studied the dynamics of relationships among three different types of agents: nomadic herdsmen, village leaders and village farmers. Nomadic herdsmen are represented by agents that need both water and grass for their cattle and who seek access to these resources granted by village leaders and farmers in return for access fees. Village leaders are represented by agents that provide herdsmen with either good or poor access to water depending on their order of arrival. Village farmers are agents who own pasture land and they may or may not permit the herdsmen to use it for cattle grazing. The authors simulate two different reasoning models for their agents: a "cost-priority" model, in which agents care only about minimizing their cost and a "friend priority" model, which is based on ideas from institutional theory, in which agents also care directly about the stability of their relationships. The authors find that the global efficiency of the cost priority model is relatively low compared with that of the friend priority model because the cost priority model tends to result in agent behavior that is less flexible.

Another interesting approach is the study of interaction mechanisms in the micro level that lead to high performing network systems. For example, Schweitzer et al. [1996] use evolutionary strategies that arise from the interaction of active Brownian agents in order to optimize the network. Schweitzer [2003] also discusses self-organization of networks and presents an agent-based model of network formation that is able to adapt the connectivity of the network to the stochastic influences in the system accordingly. More recently, the behavior of agents in innovation networks has been also investigated [König et al., 2008; Seufert and Schweitzer, 2007].

A similar research area that has attracted many researchers is coalition formation, where software agents are equipped with behavioral or cognitive models and the main goal of the agents is to find the most efficient manner in which to assign tasks to the agents and to distribute the revenues among themselves. In general, the main goal is to maximize the social welfare of the agents by means of efficient interaction protocols, strategies and coalition formation mechanisms [Larson and Sandholm, 2000; Lerman and Shehory, 2000; Shehory and Kraus, 1999; Vassileva et al., 2002]. For instance, Fiaschi and Pacini [2005] propose a model based on human capital and coalition formation. In this model, the dynamics required to form a coalition between agents depend on voluntary agreement and the payoff of the agents. This means that an agent is excluded from a coalition if the payoff of all members in the coalition may increase without having this agent in the coalition. On the other hand, an agent may join a coalition if the payoff for all members in the coalition is estimated to increase if this agent is accepted. In the same way, an agent has an incentive to enter in a coalition if his payoff increases with respect to the payoff of his actual coalition. The authors suggest in their model that the increasing returns favor the formation of coalitions, but the distributive rule of coalition makes rich agents to avoid poor agents, which leads to polarization in the resources of the agents. The authors also report that in their simulations, if the inequality in the initial distribution of resources is large, then a lower long-run growth rate of economy is observed. Kraus et al. [2004]; Shehory and Kraus [1999] present protocols and strategies for coalition formation in systems with uncertain heterogeneous information and for systems with incomplete information under time constraints. In particular, the authors investigate environments with limited computational resources and show that the compromise strategy that they propose is more stable and results in a greater increase in the social welfare of agents than non-compromise strategies.

1.4. Testbed Outline

In computer science, a *software agent* is a computational entity that is able to perceive the state of its environment and depending on this state, perform some actions in pursuit of a goal. An agent is said to be autonomous if its actions depend on the way that it processes environment information, its internal knowledge and previous experience. In the literature, there is not a clear agreement for the classification of agents (see [Wooldridge and Jennings, 1994, 1995]). For example, in the field of artificial intelligence and complex systems, one can distinguish between two types of agents: first, agents which react only to external changes (also known as “zero-intelligence agents” [Farmer et al., 2005; Gode and Sunder, 1993]) and second, agents which have a complex internal architecture (e.g. “belief-desire-intention agents”). Moreover, these agents may be immerged in multiagent systems with different purposes, for example: (i) for decision making which results from the collective cooperation between the agents,in which the main goal is to optimize resources; and (ii) for modelling, where the main goal is to explore and understand some real phenomena by means of abstracting the components and factors involved in the process, into the properties of the agents and the interaction mechanisms (for more on multiagent systems see [Russell and Norvig, 1995; Wooldridge, 2002]).

For the purpose of this thesis, we divide the classification of “zero-intelligence agents” into *reactive agents* and *experience-based agents*. This is done in order to make a clear distinction between these agents with respect to their internal complexity. Thus, we classify agents using three types:

- Reactive agents.- base their actions on simple reactions to their environment, i.e their behavior is based in functions like: $State(i) \rightarrow Action(i)$. For this, the reactive agent usually has a repository of state-action rules. An agent first performs a perception process in which it receives information about the actual state of its environment. Afterwards, in the the decision-making process, it selects an appropiate action for the current state using the set of state-action rules. Finally, an action is performed which may modify the state of the environment. A schematic diagram of a simple reactive agent is shown in Fig. 1.2. For more on these type of agents see [Schweitzer, 2003].

- Experience-based agents.- these base their actions on previous experiences, meaning that they consider previous events or previous states of the environment in order to select their current actions. Unlike a reactive agent, the experience-based agent has a history repository, where the agent saves previously observed states of the environment. This information is processed together with *previously defined* state-action rules to determine the best action to perform in the environment. In other words, the agent doesn't learn how to adapt to its environment, it only applies predefined knowledge to determine the best course of action given the previously observed states of the environment. A schematic diagram of an experience-based agent is shown in Fig. 1.2.

- Machine learning-based agents.- these have history repositories for both states of the environment and state-action rules to be mapped. Unlike experience-based agents, these agents have an *adaptive learning* module which modifies the set of "state-action" rules according to the previously observed states in the environment and the actions performed. This means that these agents are able to adapt to their environment, and learn the best actions for the different states of the environment. A schematic diagram of a machine learning-based agent is shown in Fig. 1.2. For more on these type of agents, see [Burkhard, 1993; Burkhard et al., 1998; Rao and Georgeff, 1991; Wooldridge and Jennings, 1995].

Despite these clear differences in agent architecture, it is difficult to determine what influence these properties have on the overall performance of the agents. Thus, given these different agent types, the primary concern in this PhD thesis is to study the complexity of an agent and its performance in different environments in order to answer the following questions:

- How does the (internal) complexity of an agent enhance its performance in a hard-to-predict environment?
- For which cases do agents with adaptive learning outperform reactive and experience-based agents?
- To what extent is it worthwhile to equip an agent with complex learning mechanisms instead of reactive responses to exogenous stimulus?

In order to study these questions, the scenario is expressed in an economic context, in which the main goal of an agent is to perform investments that result in larger profits. For this, an investment model has been chosen to investigate the effect of different properties, behavioral rules and learning methods on the performance of agents. A schematic general diagram for these investigations is shown in Fig. 1.2. The agent-based economical approach describes agents as investors which are given an initial budget and one goal: to maximize the amount of money over time. Thus, the budget of the agent will change over time depending on two factors: the agents' determination of the size of its investment and environmental factors. For each investment, the environment returns a payoff to the agent. Different types of environments have been chosen for these investigations: from random and stylized returns to real returns from stock market data.

Note that we can map this environment to different contexts; for example, we can think of agents that instead of trying to increase their wealth try to optimize the time they spend in performing a task. Another example is an agent with a food depot. By utilizing energy from the food depot, the agent is able to look for more food. This leads to the

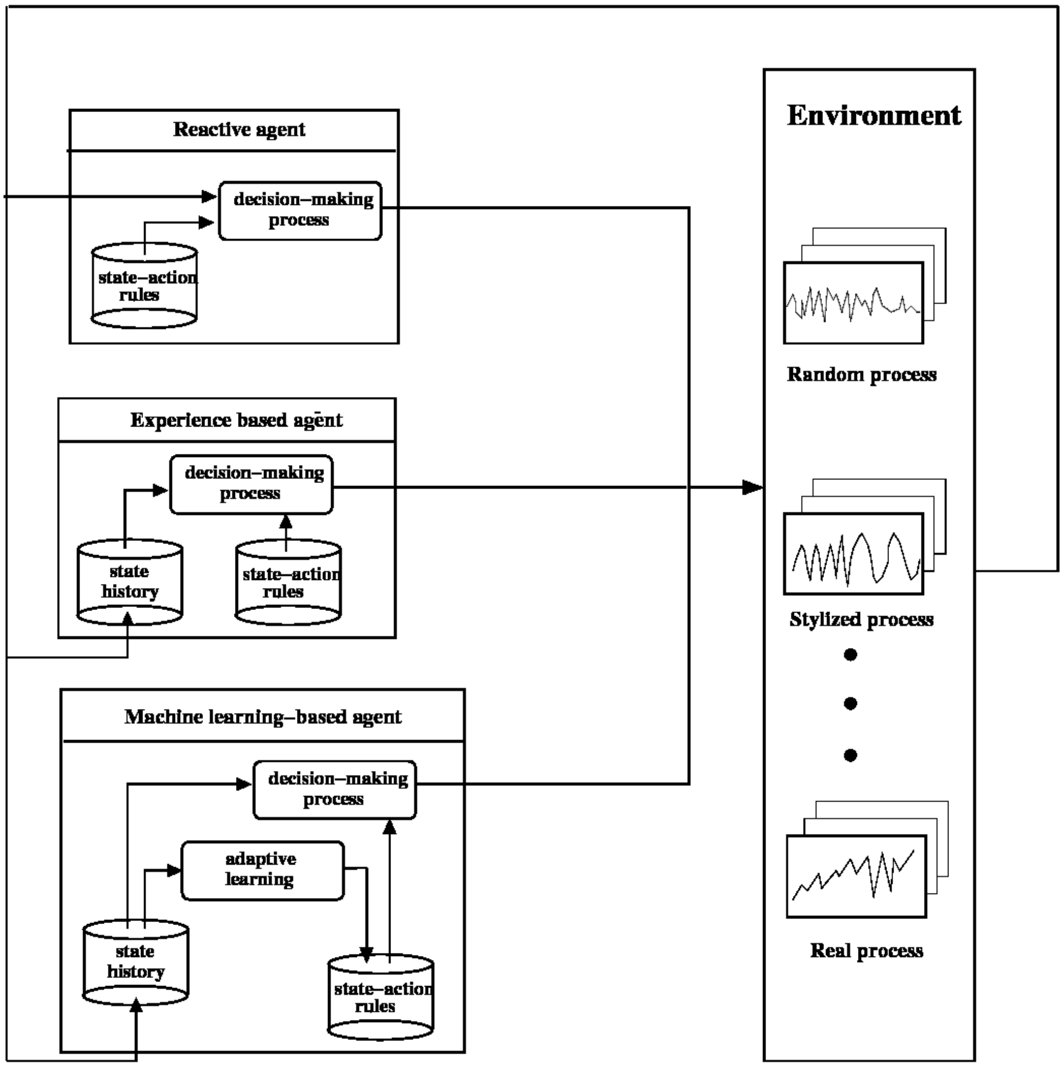


Figure 1.2.: Schematic general diagram for studying agent's complexity and performance in different environments.

optimization problem of knowing where to look for food and how much energy to use for this task [Schweitzer et al., 1998].

Because it is a necessary step in the study of more complex scenarios, an initial study of the performance of different agent architectures/investment strategies in the following simple setup is needed: each agent has a certain budget $x(t)$ and is able to invest a certain fraction of its budget in a market. The gain or loss it yields depends on the market return, or return on investment (RoI). In other words, at each time step t, the agent adjusts its investment proportion, the fraction of its budget that it is willing to invest on the market, denoted by $q(t)$, thereby controlling gains and losses resulting from the RoI, denoted by $r(t)$. The dynamics for the investment scenario can be then defined as follows:

$$x(t+1) = x(t)\left[1 + r(t)\,q(t)\right] \tag{1.1}$$

where $r(t) \in (-1, \infty)$ is the market return at time step t and $q(t)$ is the strategy of the agent which can have different degrees of complexity.

Moreover, it is assumed that only the past and current values of $r(t)$ are known to the agent; it does not know the dynamics governing future values of $r(t)$. In other words, agents observe the market through the value of $r(t)$ and, based on their analysis of a set of past $r(t)$ values, they predict future $r(t)$ values and select their behavior with respect to that market by specifying $q(t)$.

1.5. Structure of the Thesis

The research done in this PhD thesis is divided into two parts: the first part focuses on presenting the investment model and some analytical as well as numerical analysis of the dynamics of this model for fixed investment strategies in different random environments; the second part focuses on comparing the performance of different kinds of agents and investments strategies in environments that exhibit periodicities or seasonalities and an extension of the investment model for the establishment of investment networks is presented and analysed.

This PhD thesis is organized as follows: Part I presents the investment model and investigates the dynamics of the budget for fixed investment strategies. In this part, Chapter 2 presents the investment model and discusses some of its basic dynamics. Chapter 3 analyses the budget distribution and its evolution over time for different fixed investments and different random returns. Chapter 4 investigates more in detail the influence of the stochastic factors on the evolution of the budget and presents a scaling function for the most probable budget value. Chapter 5 analyses the dynamics of the budget for fixed investment strategies in returns drawn from stock market data. Part II presents different investment strategies and compares their performance in periodic environments, an extension of the investment model for the formation of investment networks is also investigated. In this part, Chapter 6 presents different agent strategies for an artificial investment scenario where the return on investment is characterized by a periodic function with different types and levels of noise. Chapter 7 compares the performance of the different strategies presented in Chapter 6. Chapter 8 extends the adaptive investment strategy based on a genetic algorithm presented in Chapter 6 and compares its performance in an environment characterized by returns with changing seasonalities. Chapter 9 extends the investment model to include formation of common investment projects. Finally, Chapter 10 presents the main contributions, conclusions and the future extensions for the investigations in this PhD thesis.

Part I.

Investment Model and Fixed Investment Strategies

2. The Investment Model

This chapter presents the investment model and a review on multiplicative random processes which basic dynamics facilitate the apprehension of the fundamental ingredients of the investment model.

2.1. Introduction

In the literature, a number of different kind of investment models can be found. Some of them are based on simple assumptions (like those based on physical models mentioned previously in Section 1.3.1), and others are based on more complex assumptions (like those based on agent-based computational economic models presented in Section 1.3.2). In this chapter, we review some of the basic dynamics that lay the fundamental ingredients for the investment model presented and discussed more in detail in the second part of this chapter.

Following Kelly [1956], assume a gambler with a given amount of money $x(t)$ at time t and he is offered the chance to bet in a lottery. To avoid losing all of his money, rather than betting a fixed amount, he prefers to always bet a fixed percentage q of his money. Let $r \in \{r^+, r^-\}$ be the results of the bet. If the gambler wins the bet, he receives r^+ times the amount he has bet, $r^+ \, q \, x(t)$ and if he loses the bet, he loses r^- times the amount that he has bet, i.e. $r^- \, q \, x(t)$.

Thus, the evolution of the gambler's budget over time t can be represented as follows:

$$x(t+1) = x(t)(1 + r\,q). \tag{2.1}$$

Assume that the lottery yields the following two results $r^+ = 1$ and $r^- = -1$, i.e. the gambler wins or loses only the amount he is betting. And assume that the gambler wins the bet with a probability p, and loses it with a probability $(1-p)$. Solving the previous difference equation using the *iterative method* [Elaydi, 1996], we find that:

$$x(t) = (1 + r\,q)^t x(0), \tag{2.2}$$

where $x(0)$ corresponds to the initial budget of the gambler.

Thus, if the gambler always bets 75% of his budget, i.e. $q = 0.75$ and has a lucky hand and always wins, i.e. $p = 1$, then the dynamics of the gambler's budget can be expressed as follows:

$$x(t) = (1.75)^t \, x(0). \tag{2.3}$$

On the other hand, if the gambler always loses the bet, the following difference equation describes the time it takes for the gambler to go bankrupt:

$$x(t) = (0.25)^t \, x(0). \tag{2.4}$$

Thus, we are interested in the following question: what percentage of his money should the gambler bet in order to maximize his long-term capital growth? If the gambler always decides to bet the same fraction, q, of his capital, then whenever he loses a bet with probability $(1-p)$, he multiplies his capital by $(1-r^{-}\,q)$, and when he wins, he multiplies his capital by $(1+r^{+}\,q)$. Thus, the average value of the logarithm of his capital after a bet is:

$$\langle \log x \rangle = (1-p)\log(1+r^{-}\,q) + p\log(1+r^{+}\,q). \tag{2.5}$$

By solving $\partial_q \langle \log x \rangle = 0$ for q it can be shown that the q for which $\langle \log x \rangle$ is maximum is:

$$q = \frac{p(r^{-}-r^{+}) - r^{-}}{r^{-}\,r^{+}}. \tag{2.6}$$

If both lottery's results are positive, then Eq. (2.6) cannot be used, as it can be seen that for unrealistic positive outcomes, the portion to invest has to be the maximum, i.e. $q = 1$.

For fair betting, $p = 0.5$, Eq. (2.6) can be simplified to:

$$q = \frac{r^{-} + r^{+}}{-2\,r^{-}r^{+}}. \tag{2.7}$$

For example, if the probability of winning the lottery is $p = 0.5$ and the results of the bet are $r^{+} = 2$ if he wins the bet and $r^{-} = -1$ if he loses the bet, then the optimal fraction of money to bet is $q = 0.25$, or 25% of his money. However, consider again the case in which the results of the bet are $r^{+} = 1$ if the gambler wins the bet and $r^{-} = -1$ if he loses the bet, i.e. a symmetrical bet or an actuarially fair lottery. For an optimal investment, $q < 0$ and assuming for the moment that it is not feasible to bet a negative portion of budget, then the optimal fraction to bet is zero percent of the money, i.e $q = 0$. Refer to the Appendix 10.2 for more details about the case in which $q < 0$. This would mean that a gambler is willing to pay money so as to avoid participating in the lottery.

Fig. 2.1 shows the optimal fraction of money q calculated using Eq. (2.7), that a gambler should bet in an actuarially fair lottery offering r^{+} and r^{-} with equal probability but with different resulting amount. It's clear that the optimal fraction increases according to the positive result of the lottery r^{+}.

Interestingly, even thought both events $r^{-} = -1, r^{+} = 1$ are equally probable, the best thing to do in such scenario is not to bet. This can be explained by means of the arithmetic and geometric mean of a series of random values. Recalling Eq. (2.3) and Eq. (2.4), the arithmetic mean of the two possible values 1.75 and 0.25 is exactly 1.0, and this would mean that the budget of the gambler should remain stable. However, the change in budget in Eq. (2.2) is expressed not as the sum of random values $(1 + r\,q)$, but as the product of them, and therefore the geometric mean must be used in this case, this corresponds to the product of these two values, i.e. 0.4375. Thus, the geometric mean of the multiplicative process corresponds to:

$$\langle \lambda(t) \rangle_{geom} = \prod_{i=1}^{t} (1 + r\,q) = (0.4375)^{t/2}, \tag{2.8}$$

and it can be seen that the gambler will eventually go bankrupt over the course of time if he decides to bet.

For the sake of clarity, the evolution of the budget is shown for the first time steps for a gambler with initial budget $x(0) = 10$ who bets the fraction $q = 0.5$ in an equally probable lottery with resulting amounts: $r^{-} = -1$ and $r^{+} = 1$. All possible budgets after betting twice in this lottery are: $x(2) = \{2.5, 7.5, 7.5, 22.5\}$, see Fig. 2.2. If we use the arithmetic

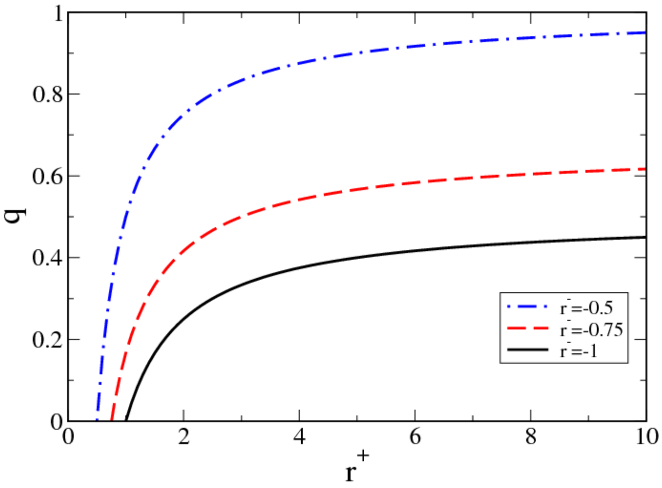


Figure 2.1.: Gambler's fraction of money, q, that leads to maximum gains, Eq. (2.7), in a actuarially fair lottery, i.e two equally probable results r^+ and r^-, but with different resulting amounts.

mean, we may find that $\langle x(2) \rangle = 10$, which corresponds to the initial budget of the gambler, however, if we use the geometric mean we find that $\langle x(2) \rangle_{geom} = 7.5$, which is the gambler's most probable budget value in this lottery.

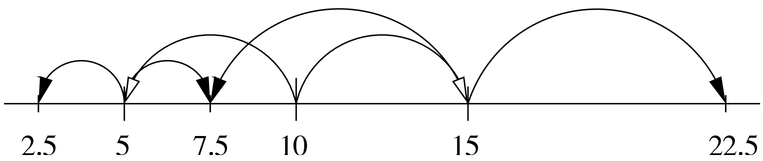


Figure 2.2.: Possible budget after betting twice in an equally probable lottery with resulting amounts: $r^- = -1$ and $r^+ = 1$.

So far, some simple betting dynamics have been introduced and the problem of bankruptcy has been discussed when the gambler iteratively bets a fixed fraction of the budget in a lottery. In the next Section, the investment model used in this thesis is presented.

2.2. The Investment Model

Previously, a simple wealth model for a gambler was presented. In this section, the investment model is introduced.

The basic ingredient of the investment model is an agent characterized by three individual variables: (i) *budget* $x(t)$, which is a measure of its "wealth" or "liquidity", (ii) *proportion of investment* $q(t)$, i.e. the fraction of budget that the agent prefers to invest in a market and (iii) *external income* $a(t)$, which can be also seen as external sources available to the agent. Thus, agent's wealth evolves over the course of time t as follows:

$$x(t+1) = x(t)\left[1 + r(t)\,q(t)\right] + a(t). \tag{2.9}$$

In this model, in time t, the agent invests a portion $q(t)x(t)$ of its total budget. This investment yields a gain or loss in the market, expressed by $r(t)$, the return on investment, *RoI*. Therefore, $r = -1$ would mean a total loss of the investment, but the gain, in principle, is not bound to a maximum. Furthermore, we assume that the market, which acts as an *environment* for the agent is not influenced by its investments. This means that instead of modeling a real market dynamics for investments, in our model we assume rather simple external dynamics for the RoI, $r(t)$, i.e. the returns are exogenous. This means that, agent's decisions change only the course of its own budget and not the process that governs the dynamics of the RoI, i.e. there is not feedback of the investment strategies on the market. In other words, we do not consider the influence that the adjusted proportion of investment has on the market return, i.e. the influence of $q(t)$ on $r(t)$. Fig. 2.3 illustrates the dynamics of the model: $r(t)$, the market return, influences the strategies agents use to adjust $q(t)$, the proportion of investment. This is a crucial assumption which makes our approach different from other attempts to model a real market dynamics, e.g. in financial markets [LeBaron, 2001; Lux and Marchesi, 2002; Raberto et al., 2003].

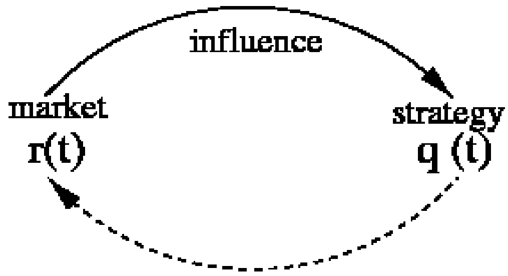


Figure 2.3.: Dynamics of the market as indicated by the market return $r(t)$ and the strategy as defined by the invested budget $q(t)$.

For simplicity, we assume for the moment that the agent invest independently in the market, i.e. there is no direct interaction with other agents. In Chapter 9, different models are presented in which interaction between agents is considered.

Furthermore, the individual agent behavior is expressed in terms of its proportion of investment, $q(t)$, which may change e.g. dependent on the agent's previous experiences, assumptions about the market dynamics or simply by trial and error. Note that Iglesias et al. [2004] refer to the proportion of investment $q(t)$ with the term *risk-aversion factor*, and the authors consider an agent that invests part of its budget in different types of fluctuating returns, saving the rest. Since $q(t)$ always represents a portion of the total budget x, it is bound to a maximum value, q_{max}, and a minimum value $q_{\min}$, in the following range:

$$0 \leq q_{\min} \leq q(t) \leq q_{max} \leq 1. \tag{2.10}$$

In the rest of this chapter and in some sections of the following chapters we consider constant parameter values for proportion of investment and income. Thus, for simplicity we may sometimes refer to these parameters with constant values: $q(t) = q$ and $a(t) = a$. Notice that Eq. 2.9 can be also formulated in terms of a quantity $I(t)$ invested at time t:

$$I(t) = x(t)\, q(t) \tag{2.11}$$

This leads to the following alternative formulation of the wealth dynamics:

$$x(t+1) = x(t) + r(t)\, I(t) + a(t) \tag{2.12}$$

in which $I(t) < x(t)$, meaning that an agent can invest only as much as it owns.

So far, the investment model has been introduced and now the dynamics of the wealth that result using Eq. (2.9) are shown and discussed for the following cases:

1. for no income and fixed proportion of investment, i.e. $a(t) = 0$, $q(t) = const.$
2. for fixed investment amount, i.e. $I(t) = I = const.$, $q(t) = I/x(t)$.
3. for positive income and fixed proportion of investment, i.e. $a(t) > 0.$ and $q(t) = const.$

In order to study the dynamics of the wealth, note that, for example, following Elaydi [1996], if we assume that the returns, the proportions of investment and the incomes all are constant, i.e. $r = r(t)$, $q = q(t)$ and $a = a(t)$ respectively, then the following closed solution can be found:

$$x(t) = \begin{cases} (1 + r\,q)^t x(0) + a\,\left[\frac{(1+r\,q)^t - 1}{(1+r\,q) - 1}\right] & \text{if } (1 + r\,q) \neq 1 \\ x(0) + a\,t & \text{if } (1 + r\,q) = 1. \end{cases} \tag{2.13}$$

Thus, if the RoI is deterministic, i.e. $r(t) = r$, the agent can use the solution in Eq. (2.13) to determine the budget at any time step. However, this task becomes more complicated if we consider that the RoI, $r(t)$, is obtained from a stochastic process, i.e. the RoI can be drawn from known probability distributions, modeled using stochastic processes or taken from real stock market data. This is shown in Fig. 2.4, where two simulations of the same dynamic with the same parameters lead to different budget values, $x(t)$, over the course of time. For these simulations, the RoI were drawn from a Binomial distribution, $r(t)\ B\{-1; 1\}$ and we assumed both proportion of investment, q, and income, a, to be constant. This implies that at every time step, the previous budget value is multiplied by two possible values $\{1 - q; 1 + q\}$ and added to the external income a. It can be seen that a particular simulation of the dynamics of $x(t)$, using Eq. (2.9), does not provide clear information about the dynamics of the agent's wealth.

Following Haan et al. [1989], if we use the iterative method for difference equations, it can be shown that the time-dependent process $x(t)$ in Eq. (2.9) can be described as follows:

$$x(t) = x(0) \prod_{i=0}^{t-1} [1 + r(i)q(i)] + a(t) \left[1 + \sum_{j=0}^{t-2} \prod_{k=j+1}^{t-1} (1 + r(k)q(k))\right]. \tag{2.14}$$

The first term on the r.h.s. corresponds to a multiplicative process described by Eq. (2.2), which can be solved for $q(i) = q$ using the geometric mean, Eq. (2.8). However, the second term on the r.h.s. is more difficult to handle even for constant values of q and a, because of the sum of multiplication of random values. Refer to Appendix 10.2, for more information regarding the analytical results for the equilibrium state of Eq. (2.9). A different approach to investigate the dynamics of a model with random variables is to simulate the dynamics

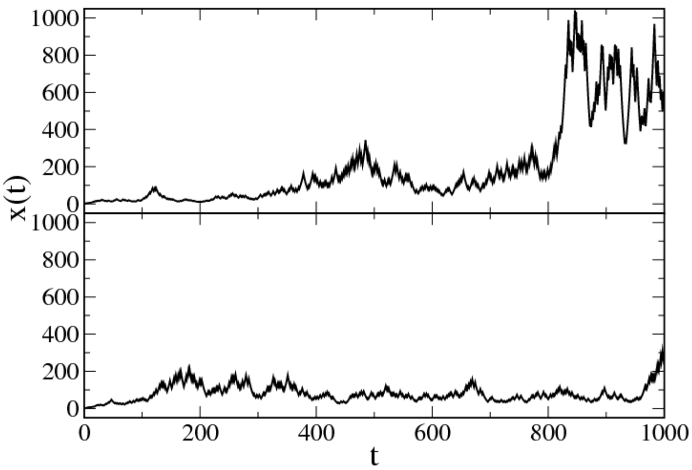


Figure 2.4.: Budget over the course of time for two independent simulations using Eq. (2.9). Parameters: $x(0) = 10$, $q = 0.1$, $a = 0.5$, $r(t)\ B{-}1; 1$ and $t = 10^3$.

of the model and to investigate the distribution of the values of the process over the course of time and for different parameter values. One method used by many researchers for these purposes is the method of Monte Carlo simulations.

Monte Carlo (MC) simulations are computational methods used to simulate the behavior of systems with random variables. They are also commonly used to compute the probability distribution of a stochastic process. In most of our investigations, we use MC simulations to obtain the probability distribution of a random process, in this case the budget of an agent for a given time step $P_x(t)$. For clarity, the Monte Carlo algorithm used in this PhD thesis to obtain the budget distribution of the agent is shown in Algorithm 1.

Algorithm 1: Monte Carlo Algorithm to obtain the probability distribution of the budget

Input: Initial budget $x(0)$, proportion of investment q, income a, return on ivnestment $r(t)$, maximum number of time steps $t_{\max}$ and number of trials N_a
Output: Probability distribution $P_x(t)$ of the budget x

```
Initialize the number of time steps t = 0
for t < t_max do
    foreach trial do
        Increase time step, t = t + 1
        Obtain the return on investment r(t)
        Update the budget x(t) using Eq. (2.9)
    end
end
Obtain the normalized histogram of the values of x(t) for all trials, i.e. P_x(t)
```

For case (i), Fig. 2.5 (left) shows the probability distribution of agent's budget, Eq. (2.9), where the agent does not receive income, $a(t) = 0$, its proportion of investment is $q = 0.1$ and the RoI is drawn from a Uniform distribution, $r(t) \sim U(-1, 1)$. For comparison an visibility reasons, the distributions are shown in a log-linear plot. As discussed in Section 2.3.1, the budget of the agent tends to zero over the course of time, because without external incomes and actuarially fair returns, the dynamics of the wealth distribution mirror those of a multiplicative random process with no repulsion from zero. This fact is also discussed in [Navarro and Schweitzer, 2003], where we propose a model for coalition formation, where agents realize common investment projects and their budget dynamics are modeled based on a multiplicative stochastic process with no additive terms.

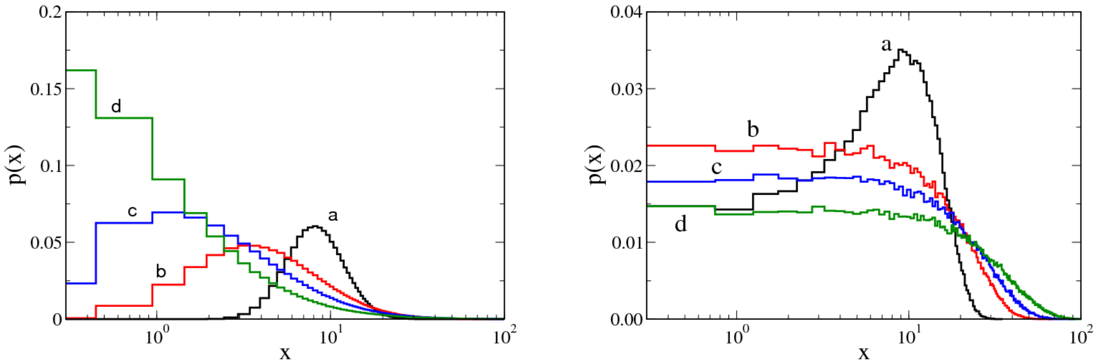


Figure 2.5.: Probability distribution of budget x following the dynamic in Eq. (2.9) for different time steps t: a) 100, b) 500, c) 1000 and d) 2000, and for: (left) case (i) $a(t) = 0$, $q(t) = 0.1$, and (right) case (ii) $I = 1$, $a = 0$. Further parameters: $x(0) = 10$, and $N_a = 10^5$.

For case (ii), if we assume that the agent always invests a fixed investment amount $I(t) = I$, this means that the fraction of budget invested at every time step is not constant and is calculated as $q(t) = I/x(t)$. Fig. 2.5 (right) shows the probability distribution of the agent's budget for $I = 1$, i.e. $q(t) = 1/x(t)$, $r(t) \sim U(-1, 1)$ and $a = 0$. For visibility reasons, the range of probability values is different to those in Fig. 2.5 (left). However, note that without income, the budget distribution of the agent tends towards zero with a large probability, though not with a large probability as for case (i). Note that the tail of the distribution in case (ii) becomes less heavy on the left over the course of time.

Note that if we increase the income by letting $a = 0.1$, Fig. 2.6 shows that $p(x)$ has a log-normal distribution that evolves to larger positive budget values over the course of time.

For case (iii), we consider a positive fixed income of $a(t) = 0.1$ and a fixed proportion of investment $q(t) = 0.1$. Fig. 2.7 shows the evolution of the budget distribution over the course of time. It can be seen that the probability distribution of the budget increases to larger positive values. Interestingly, it can also be seen that the tail of the distribution shifts to larger positive values over time.

Note that in the previous simulation experiments for the dynamic in Eq. 2.9 for random RoI with equally probable positive and negative outcomes: case (i) results in an agent going bankrupt in the long run, Fig. 2.5 (left); case (ii) is unrealistic because the agent is constraint to invest always the same fixed amount of money whatever its budget, for example an agent

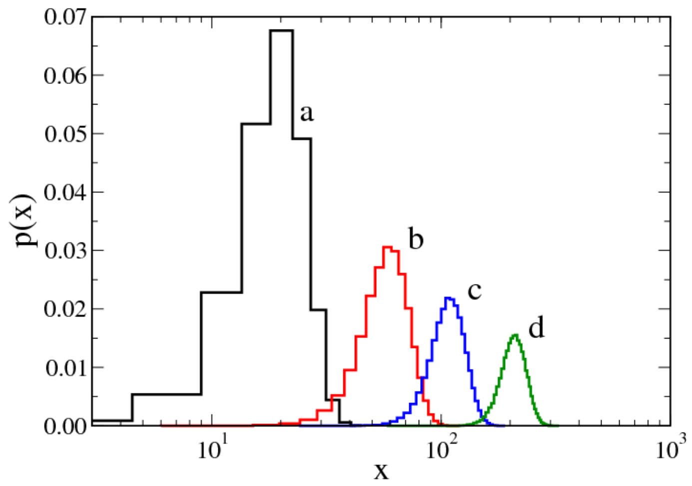


Figure 2.6.: Probability distribution of the budget x following the dynamic in Eq. (2.9) case (ii). For time steps t: a) 100, b) 500 c) 1000 and d) 2000. Further parameters: $x(0) = 10$, $I = 1$, $a = 0.1$ and $N_a = 10^5$.

investing always the amount $I(t) = 1$ even if its wealth is $x(t) = 10000$, Fig. 2.5 (right). Note also that in the long run the distribution of wealth suggests an over-enrichment of the agent which is mainly caused by the accumulation of incomes, see Fig. 2.6; case (iii) is a more real case where an agent invests proportionally to its budget and the probability distributions of the budget indicate that in the long run, the agent will probably neither go bankrupt nor go rich, see Fig. 2.7.

2.3. Understanding the Dynamics of the Investment Model

In the previous section, the investment model was presented and it was shown that for some cases, the wealth of an agent approaches zero or infinite positive values in the long run. In this section, we try to gain more insight into this fact. Note that we refer to x as the wealth of an agent, investor or gambler pursuing a bet or an investment, and these terms are used interchangeably. The process x may also be interpreted as agent's energy, resources, expected life time, etc., and we are particularly interested in the ways in which this changes over the course of time depending on the environment of the agent and its strategy of action. In the literature, we found that the investment model proposed in the previous section is related to the theory of multiplicative random processes and in the following we review some of the main results in this area.

2.3.1. Multiplicative Random Processes

Assuming again case (i) in Section 2.2, i.e. Eq. 2.9 with no incomes, $a(t) = 0$, note that the process $x(t)$ increases or decreases in value over the course of time by means of a

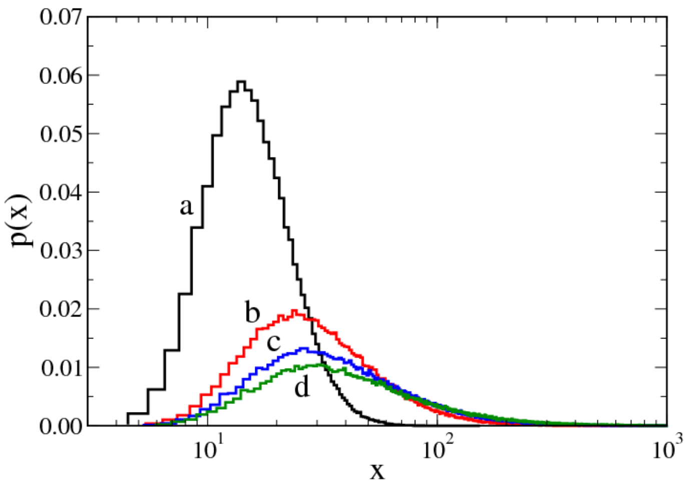


Figure 2.7.: Probability distribution of the budget x following dynamic in Eq. (2.9) (case iii). For time steps t: a) 100, b) 500 c) 1000 and d) 2000. Further parameters: $x(0) = 10$, $q(t) = 0.1$, $a = 0.1$ and $N_a = 10^5$.

multiplicative random value $\lambda(t)$ as follows:

$$x(t+1) = \lambda(t)x(t), \tag{2.15}$$

where $\lambda(t)$ corresponds to a non-negative stochastic coefficient with probability distribution $p(\lambda)$. In physical terms, this stochastic variable represents a dissipation onf x if $\lambda(t) < 1$ or amplification of x if $\lambda(t) > 1$.

In terms of the initial value $x(0)$, for fixed $\lambda \neq 1$, the solution to Eq. (2.15) is:

$$x(t) \;=\; \lambda^t \, x(0). \tag{2.16}$$

By iteration, Eq. (2.15) takes the following form:

$$x(t+1) = x(0) \left(\prod_{i=1}^{t} \lambda(i) \right). \tag{2.17}$$

Notice that if the logarithm is taken on both sides of Eq. (2.17), the logarithm of the distribution of products in Eq.(2.17) for large t, can be approximated by the sum of t random variables. That means the logarithm of the probability of the budget, $\log p(x)$, follows a Gaussian distribution. In other words, the distribution of the product has a log-normal distribution as shown by Redner [1990]. Moreover, Sornette and Cont [1997] express the distribution of x as follows,

$$P(x) = \frac{1}{\sqrt{2\,\pi\,D\,t}} \frac{1}{x} \exp\left(-\frac{1}{2\,D\,t} (\log x - \langle \log \lambda \rangle \; t)^2 \right), \tag{2.18}$$

where:

$$\langle \log \lambda \rangle = \int_0^\infty \log \lambda \, p(\lambda) d\lambda \tag{2.19}$$

$$D = \left\langle (\log \lambda)^2 \right\rangle - \langle \log \lambda \rangle^2 . \tag{2.20}$$

Note that if $\langle \log \lambda \rangle > 0$, then $x(t) \to \infty$, but if $\langle \log \lambda \rangle < 0$, then $x(t) \to 0$. That means if the random process λ is biased and approaches to infinity, then $x(t)$ will be distributed by Eq. 2.18, otherwise it will shrink to zero.

Assuming that the coefficient $\lambda(t)$ is drawn randomly from a uniform distribution, $\lambda(t) \sim U(a, b)$, then the quantities $\langle \log \lambda \rangle$, $\langle (\log \lambda^2 \rangle$ and D can be calculated as follows:

$$\langle \log \lambda \rangle = \int_a^b \log \lambda \, p(\lambda) d\lambda \tag{2.21}$$

$$= \frac{1}{b-a} \int_a^b \log \lambda d\lambda = \frac{1}{b-a} \left[\lambda \, \log \lambda - \lambda \right]_a^b$$

$$= \frac{b \log b - a \log a - b + a}{b-a} \tag{2.22}$$

$$\left\langle (\log \lambda)^2 \right\rangle = \int_a^b (\log \lambda)^2 \, p(\lambda) d\lambda \tag{2.23}$$

$$= \frac{2\, a \log a - a \log a^2 - 2\, b \log b + b \log b^2 - 2\, a + 2\, b}{b-a} \tag{2.24}$$

$$D = \frac{(a-b)^2 - a\, b(\log a - \log b)^2}{(a-b)^2} . \tag{2.25}$$

These previous equations can be used to elucidate some of the dynamics of the stochastic process $x(t)$. For example, for $\lambda(t) \sim U(1, 2)$ it can be shown that $\langle \log \lambda \rangle = 0.3862$, which means that the process has a drift to the right, i.e. to larger positive values. On the other hand, if $\lambda(t) \sim U(0.5, 1.5)$ it can be shown that $\langle \log \lambda \rangle = -0.04$, which means a drift to the left, i.e. to zero.

These previous theoretical results can be confirmed by comparing them with results that can be obtained using computer simulations. For this, Monte Carlo simulations were used to simulate the dynamics of Eq.(2.15) in order to obtain the probability distribution of the process for different time steps. Fig. 2.8 shows the evolution of the probability distributions for this multiplicative random process for both simulations and theoretical distributions of Eq. (2.18). For these, two different λ are considered: (left) $\lambda(t) \sim U(1, 2)$ showing a drift to $+\infty$ values; and (right) $\lambda(t) \sim U(0.5, 1.5)$ showing a drift to $-\infty$, i.e. for $t \to \infty$, $x(t) \to 0$.

Furthermore, Sornette and Cont [1997] show that the probability distribution of the maximum value reached, $p(x_{\max})$, also exhibits a power law distribution on the tail. This is confirmed in Fig. 2.9 (left), where $p(x_{\max})$ is shown for two different time steps: $t = \{100, 10^4\}$. It is clear that both have approximately the same power law distribution on the tail. This means that for large t, the distribution of the maximum values reached in the process reaches a stationary distribution. This phenomenon is frequently referred as an extreme form of temporally intermittent bursting, also called on-off intermittency [Venkataramani et al., 1996].

The next section shows what would happen if an extra ingredient is added to this multiplicative stochastic model.

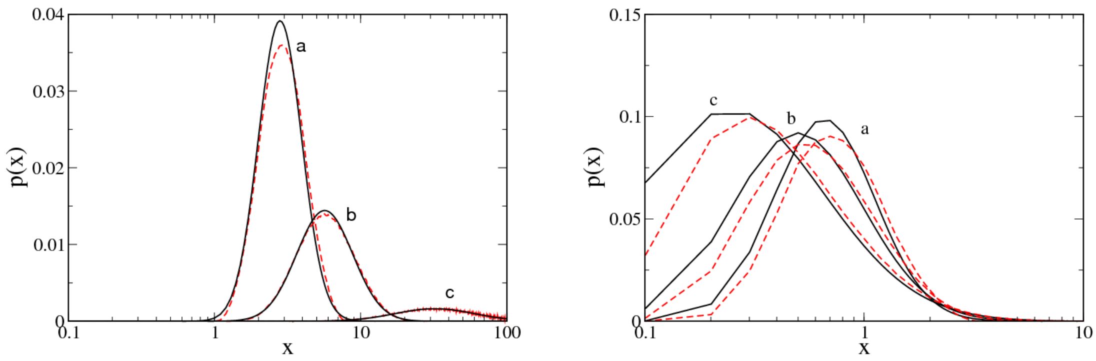


Figure 2.8.: Probability distribution of x, Eq.(2.15), theoretical (bold), simulation (dashed) results, for a) $t = 3$, b) $t = 5$ and c) $t = 10$. For (left) $\lambda(t) \sim U(1, 2)$, and (right) $\lambda(t) \sim U(0.5, 1.5)$. Further parameters: $x(0) = 1$ and $N_a = 10^5$ trials.

2.3.2. Multiplicative Random Process with Repulsion at the Origin

Imagine the following situations: (i) the government or other organization helps people by preventing them for going bankrupt or (ii) an investor borrows money from friends or from the bank in order to continue investing in, for example, the stock market. In other words, the question now is the following: what if investors are not allowed to go bankrupt?

Levy and Solomon [1996] show that if the boundary constraint $x(t) > x_{min}$ is added to Eq. (2.15), and if x has a drift towards $-\infty$ (i.e. $\langle \log \lambda \rangle < 0$), then there will be a compensation between the negative drift, which leads $x \to 0$ and a repulsion off the barrier, $x_{min} > 0$, which will cause $x \to +\infty$. This repeated back and forth motion leads x to be not anymore distributed according to a log-normal, but according to a power law,

$$P(x) \sim x^{-(1+\mu)}, \tag{2.26}$$

where the exponent μ is determined solving,

$$\int_0^{+\infty} \lambda^{\mu}\, p(\lambda)\, d\lambda = 1. \tag{2.27}$$

The two ingredients needed to obtain power law distributions are then: (i) a barrier or a boundary constraint that forces $x(t)$ to remain larger than a minimum value $x_{min} > 0$ and (ii) the random process fulfills the condition $\langle \log \lambda \rangle < 0$, meaning that the stochastic coefficient $\lambda(t)$ has a drift towards the left. Thus, as the process starts to evolve over the course of time, it follows a random walk with a tendency towards zero values, but it is repelled from zero by the barrier at $x_{\min}$, leading to a power-law distribution of the process.

Moreover, as described by Sornette and Cont [1997], $x(t)$ reaches a steady-state distribution for $t \to +\infty$ following a power-law in the tail. The exponent of the power law can be calculated by solving Eq.(2.27). For example for $\lambda(t) \sim U(a, b)$, the exponent μ can be

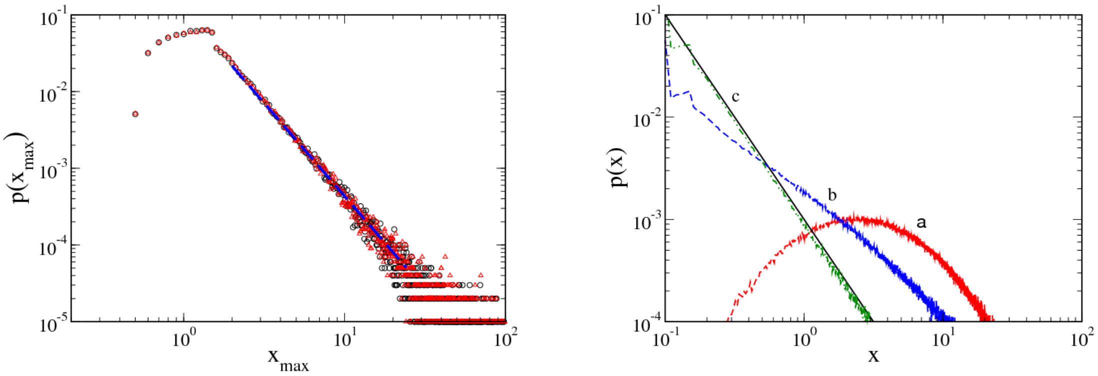


Figure 2.9.: Probability distribution of x, using Eq. (2.15) for: (left) maximum values reached, $x_{\max}$, using $\lambda \in U(0.48, 1.48)$. Simulations performed for 2 different number of time steps showing the same power law exponent $\mu \approx 1.42$: (circles) $t = 100$ and (triangles) $t = 10^4$; (right) x with boundary constraint $x_{min} = 0.1$ (Section 2.3.2) and $\lambda(t) \sim U(0.5, 1.5)$, for: a) $t = 10$, b) $t = 50$ and c) $t = 500$. Further parameters as in Fig. 2.8.

found by solving:

$$\int_a^b \lambda^\mu p(\lambda) d\lambda = 1 \tag{2.28}$$

$$\int_a^b \lambda^\mu \frac{1}{b-a} d\lambda = 1$$

$$\frac{1}{b-a}\left[\frac{\lambda^{\mu+1}}{\mu+1}\right]_a^b = 1$$

$$\frac{b^{\mu+1} - a^{\mu+1}}{(b-a)(\mu+1)} = 1, \tag{2.29}$$

where the exponent μ can be obtained using for example, the well known Newton-Rhapson method for finding approximations of the roots of a real-valued function. Another way to determine the exponent is by adjusting a straight line to the power law distribution in a log-log plot, so that the slope of the line corresponds to the exponent of the power law.

To gain more insight into the dynamics, Monte Carlo simulations were performed at the individual level and compared with theoretical values. Fig. 2.9 (right), shows the probability distributions for different time steps a) $t = 10$, b) $t = 50$ and c) $t = 500$. It is clear that when $t = 10$, the probability distribution still has a log-normal distribution, but as $t \to \infty$, the distribution of the process converges to a distribution with a power-law on the tail, Eq. (2.26) with exponent $\mu = 1$. Interestingly, the steady state is reached relatively fast, in this case after $t = 500$. In addition to the properties already mentioned, it is also of note that the initial value of the process $x(0)$ does not influence the exponent μ of the power law distribution.

In summary, if the barrier $x(t) > x_{min}$ is considered in the investment model, Eq. (2.9), in the long run, investors will not go bankrupt.

2.3.3. Multiplicative and Additive Random Process

In real situations, investors are immersed in an environment where their profits or losses depend also on external factors, which can be seen as random factors. In this section, we consider the case where besides the profits or losses due to the RoI, the investors are influenced by external additive incomes. In terms of random processes, this scenario can be modeled by a random multiplicative process with an additive term as follows:

$$x(t+1) = \lambda(t)x(t) + a(t), \tag{2.30}$$

where $a(t)$ can be a constant or stochastic term, that repels $x(t)$ away from zero. Following Elaydi [1996]. If we assume constant parameter values in Eq. (2.30), i.e. $\lambda(t) = \lambda$ and $a(t) = a$, the solution for this non-homogeneous multiplicative additive difference equation is given by:

$$x(t) = \begin{cases} \lambda^t x(0) + a\left[\frac{\lambda^t - 1}{\lambda - 1}\right] & \text{if } \lambda \neq 1 \\ x(0) + a(t-1) & \text{if } \lambda = 1. \end{cases} \tag{2.31}$$

This solution has the disadvantage that it does not consider stochastic terms.

On the other hand, if we consider the stochastic term, we find two different kind of analysis in the literature. Firstly, according to Vervaat [1979] three cases can be distinguished:

1. $\sum_{k=1}^{n} \log|\lambda_k| \ \rightarrow \ -\infty$ in distribution

2. $\log|\lambda| \ = \ 0$ with probability 1

3. $\limsup P\ [\sum_{k=1}^{n} \log|\lambda_k| \ > \ 0] \ > \ 0.$

In case (1), there is a unique solution to the stationary equation and the probability distribution is independent of the initial value. In cases (2) and (3), a special relationship between λ and a is required in order to ensure convergence to a stationary distribution. If a is constant, the stationary distribution for the first case is then given by the distribution of the following infinite series:

$$T \ = \ a\sum_{k=1}^{\infty} \lambda_1\,\lambda_2 ... \lambda_{k-1} \ = \ a\sum_{k=1}^{\infty}\prod_{j=1}^{k-1} \lambda_j. \tag{2.32}$$

It can be seen that the formula for the moments of the limit distribution shows that the mean, were it defined, should be:

$$\langle x \rangle = \frac{a}{1 - \langle \lambda \rangle}, \tag{2.33}$$

which diverges for $\langle \lambda \rangle \ = \ 1$. However, recalling the dynamics of the investment model, Eq. (2.9), it can be shown for $\lambda = 1 + r\,q$ that if the returns are symmetrical, this yields $\langle \lambda \rangle = 1$ and consequently a finite mean value cannot be expected.

Moreover, Kesten [1973] and Sornette [1998a] show that if $\lambda(t)$ is a random process with $\langle \log \lambda \rangle < 0$ and if $a(t)$ is an additive term that ensures $x(t)$ will be pushed out from a zero value, then the probability distribution of x has a power law on the tail of the form shown in Eq.(2.26). The exponent μ can be determined by solving $\langle \lambda^{\mu} \rangle = 1$ as in Eq. (2.27), see [Takayasu et al., 1997]. Note that there exists a similar behavior between a multiplicative process with a barrier and a multiplicative process with an additive term. In the latter case, the additive term $a(t)$ acts as the barrier term $x_{\min}$ producing a repulsion from zero.

Moreover, Takayasu et al. [1997] show a similar result found by taking the average over the square of Eq.(2.30). Assuming that $\langle\lambda^2\rangle$ and $\langle a^2\rangle$ are constants and $\langle\lambda^2\rangle < 1$, the authors find the following stationary solution:

$$\left\langle x^2 \right\rangle = \frac{\langle a^2\rangle}{1-\langle\lambda^2\rangle}; \tag{2.34}$$

however, the authors refer to the thermal equilibrium case, for which it is required that $\langle x^2\rangle$ be proportional to the temperature, which means that $\langle a^2\rangle$ and $\langle\lambda^2\rangle$ cannot be independent (see fluctuation-dissipation theorem [Reichl, 1980]).

Sornette [1998a] shows that if the finite difference in Eq.(2.30) is rested and divided by $x(t)$ we obtain the following equation:

$$\frac{x(t+1)-x(t)}{x(t)} = \frac{a(t)}{x(t)} + \lambda(t) - 1. \tag{2.35}$$

If the finite difference $\frac{x(t+1)-x(t)}{x(t)}$ is approximated to $\frac{\mathrm{d}\,\log x}{\mathrm{d}\,t}$. And assuming that $w = \log x$, the following over-damped Langevin equation can be obtained:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = a(t)e^{-w} - |v| + \eta(t), \tag{2.36}$$

where:

$$v = \langle\lambda\rangle - 1 \simeq \langle\log\lambda\rangle \tag{2.37}$$

$$\left\langle \eta^2 \right\rangle = \left\langle \lambda^2 \right\rangle - \langle\lambda\rangle^2 . \tag{2.38}$$

It can be seen that the first term on the right side produces a repulsion of x from zero, whereas the second and third term correspond to the mean of the multiplicative process and a purely fluctuating term respectively. In moving from the single stochastic realizations of $\omega(t)$ to the probability density $P(\omega, t)$, it was shown by Sornette and Cont [1997] that the following Fokker-Planck equation can be derived:

$$\begin{aligned}\frac{\partial P(w,t)}{\partial t} &= a(t)\, e^{-w}\, P(w,t) - \left(\langle\log\lambda\rangle + a(t)\, e^{-w}\right) \frac{\partial P(w,t)}{\partial w} \\ &\quad + \left(\left\langle \log{(\lambda)^2} \right\rangle - \langle\log\lambda\rangle^2\right) \frac{1}{2}\frac{\partial^2 P(w,t)}{\partial w^2}\end{aligned} \tag{2.39}$$

where the first term produces a decay on w, the second term indicates the drift of the process and the third is a diffusion term. Note that this equation is different from the original one first proposed by Sornette and Cont [1997] where the factor 2 in the diffusion term was missing. Moreover, a generalization of these multiplicative random processes is shown by Sornette [1998a], in which the authors show that the following dynamic also yields to power law distributions:

$$x(t+1) = e^{f(x(t),\lambda(t),a(t)..)}\lambda(t)x(t), \tag{2.40}$$

where $f(\cdot) \to 0$ for large $x(t)$ and $f(\cdot) \to \infty$ for $x(t) \to 0$.

Consider now Eq.(2.15), if $x(t) > x_{min}$ then $f(\cdot) = 0$ and if $x(t) < x_{min}$ then

$$f(\cdot) = \log\left(\frac{x_{min}}{\lambda(t)x(t)}\right), \tag{2.41}$$

whereas for Eq.(2.30),

$$f(\cdot) = \log\left(1 + \frac{a(t)}{\lambda(t)x(t)}\right). \tag{2.42}$$

To gain more insight into the dynamics of this model, see Fig. 2.10 (left) which shows the results of Monte Carlo simulations against theoretical values for the probability distribution of x for Eq.(2.30). Computer simulations were performed using $\lambda(t) \sim U(0.5, 1.5)$ for the random multiplicative process. Compared with the theoretical value of the exponent $\mu = 1.0$, we observe that the process has a power law in the tail of the distribution. The influence of the random multiplicative process is shown for two cases: a drift towards the left using $\lambda(t) \sim U(0.48, 1.48)$ (circles); the theoretical value of the exponent is $\mu = 1.4673$ and a drift towards the right using $\lambda(t) \sim U(0.52, 1.52)$ (triangles); the theoretical value of the exponent in the tail of the distribution is $\mu = 0$. We observe that the slope of the power law increases if the drift towards the left increases and that the slope decreases if the drift towards the right increases.

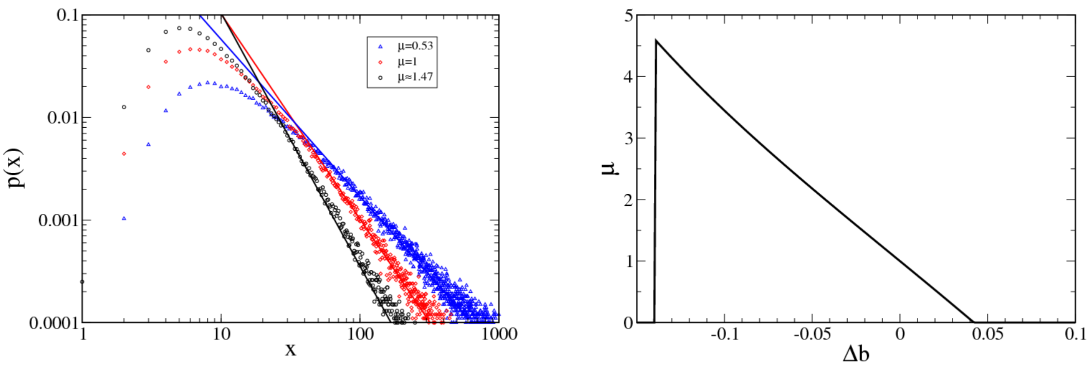


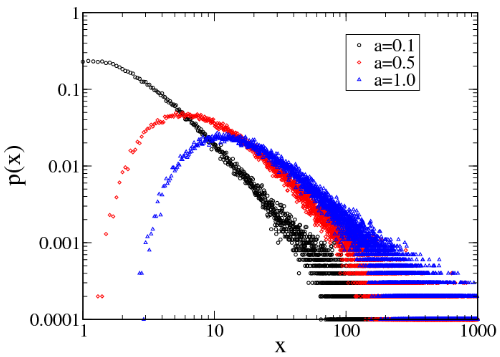
Figure 2.10.: (left) Probability distribution of x for Eq.(2.30) using Monte Carlo simulations for $N_a = 10^5$ trials. Simulations using $\lambda(t) \sim U(0.48, 1.48)$ ($\circ$), $\lambda(t) \sim U(0.5, 1.5)$ ($\diamond$) and $\lambda(t) \sim U(0.52, 1.52)$ are comparable to the theoretical exponents of the power law distributions $\mu = 1.4673$, $\mu = 1.0$ and $\mu = 0.5306$ respectively. (right) Power law exponent value μ obtained for different multiplicative random processes with $\lambda(t) \sim U(0.5+\triangle b, 1.5+\triangle b)$ where $\triangle b \in (-0.15, 0.5)$ for an incremental step of $\triangle b$ of 0.01. Further parameters (for both figures): $a(t) \sim U(0, 1)$ and $x(0) = 1$.

With regards to the influence of the random variable in the process, the following questions are of special interest: how much can we disturb the process by means of random influence? Is there a boundary for randomness where the random influence becomes strong enough to make considerable change in qualitative or quantitative behavior? To answer these questions, the influence of the multiplicative and the additive random terms has been analyzed.

Fig. 2.10 (right) shows the theoretical results of the influence of the multiplicative term or drift term $\lambda(t)$ on the exponent μ of the power law distribution. The corresponding exponents μ were obtained by finding the roots of Eq. (2.29). The graphic shows the different exponents μ obtained for uniform increasing variations in the range of values for the random variable $\lambda(t)$ from $\lambda(t) \sim U(0.35, 1.35)$ to $\lambda(t) \sim U(0.75, 1.65)$. It can be seen that the slope emerges when $\lambda(t) \sim U(0.36, 1.36)$ and then decays linearly as the drift approaches to $-\infty$.

Following Sornette and Cont [1997], a constant additive term instead of a stochastic variable is assumed. Thus, for simplicity, instead of the stochastic additive term $a(t)$ in Eq.(2.30), a constant value $a \in (0, 1)$ is assumed. Using the same multiplicative random process, $\lambda(t) \sim U(0.5, 1.5)$, we observe that the resulting probability distribution of x is roughly the same both when using a stochastic additive term $a \in U(0, 1)$ and with a constant additive term $a(t) = 0.5$; No significant difference between them is observed. The same was observed for the exponent μ of the power law in the tail of the distributions, which remains $\mu = 1$ for both cases because it depends mainly on the multiplicative random process λ.

Moreover, Fig. 2.11 (left) shows the probability distributions of x using Eq. (2.30), for three different additive values with the same multiplicative random process $\lambda(t) \sim U(0.5, 1.5)$. It is clear that the constant additive term a does not considerably change the power law distribution in the tail.



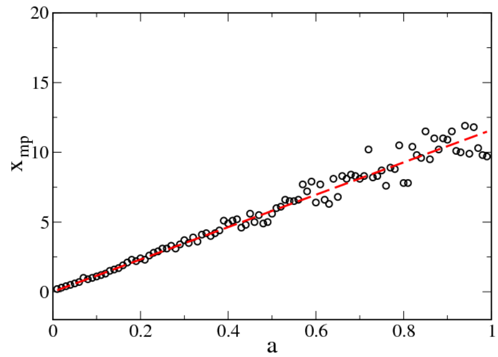


Figure 2.11.: (left) Influence of the constant additive term a on the probability distribution of x obtained from simulations. The power law on the tail of the distributions has approximately the same exponent $\mu = 1$. (right) Simulation results for the non-linearly increment of x_{mp} w.r.t. an increment in the constant additive term a, for $a = \{0.01, 0.02, .., 1\}$. The dashed line corresponds to a linear fit to the data points, Eq. (2.43). Further parameters: $\lambda(t) \sim U(0.5, 1.5)$, $x(0) = 10$ and $N_a = 10^5$ trials.

However, note that the peaks of the distribution, which correspond to the most probable value of x, denoted by x_{mp}, are modified by the additive term. It is clear that as the additive term decreases, $a \to 0$, x_{mp} also decreases, $x_{mp} \to 0$, whereas if $a \to \infty$, then $x_{mp} \to \infty$. Fig. 2.11 (right) shows the result of a number of simulations performed for different additive terms. Interestingly, for $a > 0.5$, the value of x_{mp} starts to disperse because as a increases, x takes larger positive values, increasing the diversity of possible x values. Furthermore, these results were fitted to the following linear function:

$$x_{mp} = 11.596 \pm 0.148\, a. \tag{2.43}$$

It is clear that there exists an approximately linear relation between x_{mp} and a.

2.3.4. Multiplicative Entry/Exit Random Process

In the previous random multiplicative models, a homogeneous scenario was considered for which the probability distribution of the process x follows a power law distribution if the stochastic multiplicative process has a drift towards zero and either a barrier or an additional additive term which repels x from zero.

In this section, a heterogeneous scenario is considered that yields power law distributions. In the literature, these models are usually referred as multiplicative processes with entry/exit dynamics (see [Blank and Solomon, 2000; Richiardi, 2004]).

Consider a multi-agent system of N agents, in which agent i possesses for example, a budget described by the variable $x_i(t)$. Assume that initially $x_i(0) \geq x_{\min}$, for $x_{\min} = \text{const}$ and $x_{\min} > 0$. The dynamics of $x_i(t)$ evolve over the course of time by,

$$x_i(t) = \lambda(t)\, x_i(t), \tag{2.44}$$

where $\lambda(t)$ is a stochastic process which determines how the budget of agent i changes at every time step t. Each time that $x_i(t) < x_{\min}$ the agent i is replaced by a new agent with $x_i(t) \geq x_{\min}$. Thus, in terms of wealth, if an agent i goes to bankrupt, i.e. agent's budget x_i is less than a minimum value, $x_{\min}$, it will be replaced by a new and more wealthy agent, which starts with an initial budget $x(0) > x_{\min}$. In other words, using these entry/exit dynamics, the system gets rid of agents that perform poorly. Note again that for simplicity we can think of x_i as agent's energy, wealth, expected life, resources, etc., and λ describes the way in which these change over the course of time. Furthermore, the actions of agent i may or may not affect λ, depending on the formulation of the problem.

To gain more insight into the dynamics, we performed some Monte Carlo simulations for Eq. 2.44 to obtain the probability distribution of the budget of a number of agents. Fig. 2.12 (left) shows the probability distribution of the budget for different $x_{\min}$ values. It is clear that, as asserted by Blank and Solomon [2000], this model yields distributions with a power law in the tail. For reference reasons, a power law with $\mu = 1$ (dashed line) is also plotted. We observe that the slope in the tail of the wealth distribution approximates the expected theoretical value in Eq. (2.29). Thus, the probability distribution of the budget also has a power law decay in the tail as in Eq. (2.26).

Fig. 2.12 (right) shows the probability distribution of the budget $p(x)$ after $t = 10^4$ time steps for $N = 10^5$ trials with an initial budget $x_i(0) = 1$ and $x_{\min} = 10^{-10}$. A power law, Eq. (2.26), with $\mu = 0$ is represented with a dashed-line. Notice that in Eq. (2.29), if the range of λ is symmetrical around the unit, i.e. $b - a = 1$, then Eq. (2.29) has only two possible solutions: $\mu = 0$ and $\mu = 1$. This fact is shown in both plots in Fig. 2.12. It would be interesting to investigate the critical value of $x_{\min}$ that leads to a change in the slope of the power law shown in Fig. 2.12 in more detail, however, this is beyond the purposes of this PhD thesis and therefore is left for further work. Now, observe in Fig. 2.12 (right), that for $x = 1$ the probability $p(X = 1)$ is larger than other probability values. This occurs because if the budget of an agent is less than $x_{\min}$, this agent is replaced by a new agent with initial budget $x(0) = 1$, leading to a larger probable value in the distribution. This outlier-effect can be eliminated by allowing the agent that has been replaced to start with an initial budget that is randomly drawn from a range of possible initial budget values.

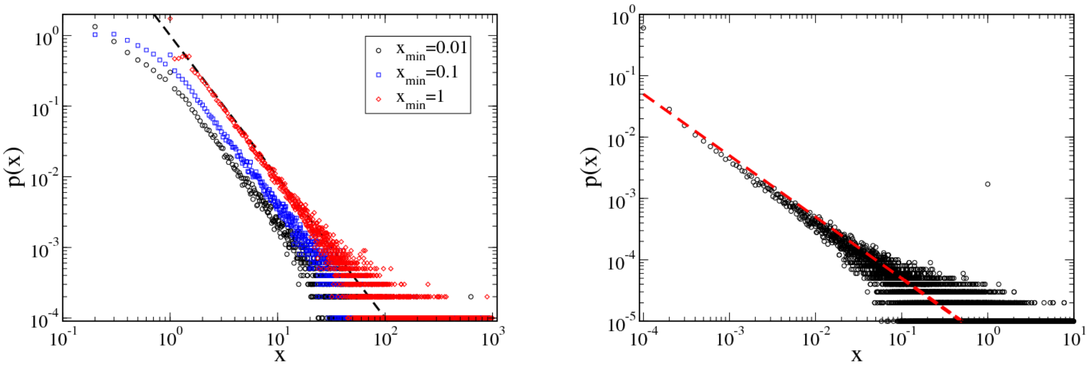


Figure 2.12.: Probability distributions of the budget x using Eq. (2.44) for: (left) different $x_{\min}$ values; (right) $x_{\min} = 10^{-10}$. Both plots present a power law in the tail (dashed-lines) with $\mu = 1$ and $\mu = 0$ respectively (see Eq. (2.26)). Further parameters: $x(0) = 1$, $\lambda(t) \sim U(0.5, 1.5)$ and $N_a = 10^5$ trials.

It is also interesting to know the probability distribution of the number of agents that are substituted during a simulation, $p(N_{\text{sub}})$. For this, Fig. 2.13 shows the histogram that was obtained from the results of a number of simulations of the dynamics in Eq. (2.44) for $N_a = 10^4$ trials and for $t = 1000$ time steps. We can see that the number of substituted agents during a simulation has a Gaussian probability distribution.

Finally, it would be interesting to investigate the influence of the different parameters on the distribution of substituted agents, for example, for $\lambda(t)$ drawn from different interval values or different distributions. Moreover, it would be also interesting to find a relationship between the initial budget $x(0)$, the minimum budget value $x_{\min}$ and λ with respect to the distribution of N_{sub}, however, these are beyond the purposes of this PhD thesis and therefore are left for further work.

2.4. Summary

In this chapter, the investment model is presented; this model is based on a multiplicative stochastic process with an additive term. This approach also presents a different point of view for investment strategies, where the agent (the investor) interacts with its environment, but does not modify his environment with its actions. In other words, the agent analyzes its environment and tries to predict it, but its decisions only change the course of its own properties. A review of multiplicative stochastic processes is also presented to provide more insight into the dynamics of the investment model.

In the following chapters, different *strategies for controlling the proportion of investment*, expressed by various forms of $q(t)$, are presented and investigated for different kinds of RoI.

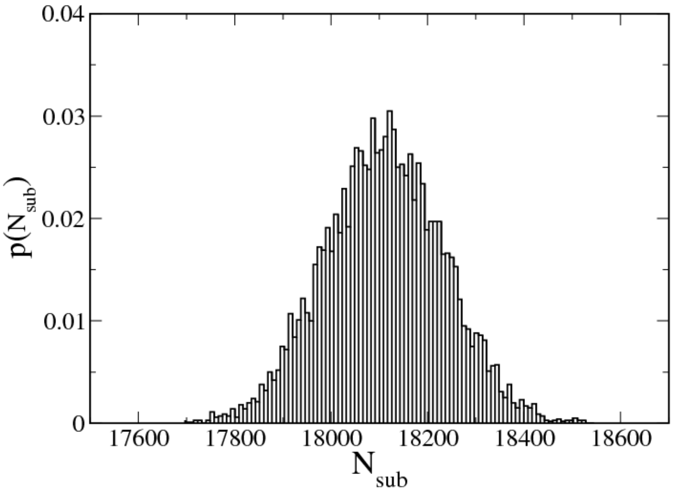


Figure 2.13.: Probability distribution of number of substituted agents, N_{sub}, on a model with entry/exit dynamics. Parameters: $x(0) = 1$, $\lambda(t) \sim U(0.5, 1.5)$, $a(t) \sim U(0, 1)$, $t = 10^3$ and $N_a = 10^3$.

3. Budget Evolution for Constant Proportions of Investment

In this chapter the budget distribution and its evolution over time is analysed for different fixed investments and different random returns.

3.1. Introduction

So far, Chapter 1, gave some motivation for the problem of finding proper investment strategies. Chapter 2 presented the investment model, Equation (2.9), and different multiplicative processes related to the investment model.

The main goal of this chapter is to investigate the dynamic of the budget of an agent for different fixed proportions of investment, $q(t) = q(0)$, as well as to investigate the influence of other parameters like the external sources a and random returns drawn from different probability distributions.

This means that over the course of time, the agent always invests the same initial proportion of investment. We note that a typical scenario used to study investment strategies is to allow an agent to choose between investing in a risk-free asset or in a risky asset (see [Pratt, 1964; Tobin, 1958]). In our investment scenario, the agent invests a constant proportion of the budget q in the risky asset while keeping the rest in a risk-free asset. Usually, economists' approach to the decision-making problem of investing in risky assets or risk-free assets is addressed in terms of utility functions. For the sake of simplicity, in this chapter, we start our investigations by considering fixed investments; however, in the Appendix 10.2 we show the relationship between the utility of wealth and constant proportion of investment.

In Fig. 2.4, the results of two simulations of Eq. (2.9) were showed, which were run independently with the same parameter values, leading to different dynamics of the budget over the course of time. Thus, in order to find out more about the dynamics of the wealth of an agent described by the stochastic process in Eq. (2.9), in this chapter, the influence of the random RoI in the evolution of the wealth is analyzed using Monte Carlo simulations. For the Monte Carlo simulations, Algorithm 1 in Section 2.2 is used to obtain the probability distribution of the budget of a number of simulations or trials, N_a, where each simulation is a replication of the dynamic in Eq. (2.9) for the same initial parameter values. The stochastic factor is given by the RoI which is at every time step drawn randomly from a known probability distribution or obtained from a given random process. Actually, the fixed investment strategy may be seen less as a strategy and more as an attitude towards the risk of the investment, in which the investor neither looks at the previous returns nor his budget; he only assumes a fixed proportion of investment value and uses it throughout the whole simulation. However, we use the term “strategy” in an attempt to generalize the range of possible approaches and methods an agent can use in order to decide how much to invest. Moreover, this strategy serves as a reference strategy in Chapter 7, where it is used to determine whether other more complex strategies are able to outperform this simple strategy.

Thus, in this chapter, we investigate the influence of different fixed proportions of investment and fixed incomes on the budget distribution and its evolution over the course of time. In Section 3.2, we use simulations to analyze the dynamics of the budget for different constant proportions of investment and different income values. Finally, in Section 3.3, we show an analytical solution for the stationary probability distribution of the budget, which corroborates the simulation results.

3.2. Simulation Experiments

In this section, computer simulations of Eq. (2.9) are presented for different distributions of $r(t)$. For our simulations, we assumed that the initial budget of the agent is $x(0) = 10$ and that the proportion of investment and the income are kept constant during each simulation, i.e. $q(t) = q$ and $a(t) = a$.

Thus, to elucidate the dynamics of agent's budget a number of computer simulations are performed to gather the evolution of the probability distribution of the budget $p(x,t)$ and the average of the most probable budget value $\langle x_{mp}(t)\rangle$. First, in order to determine the distribution of the budget of an agent, the dynamic in Eq. (2.9) was simulated for a number of time steps t, yielding $x(t)$, i.e. the budget of the agent at time step t. Because of the random RoI, we may need to repeat the previous experiment a number of trials N_a, which may lead to different values of $x(t)$. After a large number of trials, the histogram of these values eventually yields to the probability distribution of the budget $p(x,t)$ for time step t. We are also interested in the most probable budget value $x_{mp}(t)$ which, once we have obtained the $p(x,t)$, corresponds to the peak of the distribution. Because of the random component in the model, we obtain a number N_s of distributions $p(x,t)$ and afterwards, we calculate the average of the distribution's peaks to obtain $\langle x_{mp}(t)\rangle$.

For almost all the computer experiments performed in this section, a number of $N_a = 10^4$ independent trials of the dynamic in Eq. (2.9) were performed to obtain the budget distribution and we obtained $N_s = 10$ budget distributions to find each $\langle x_{mp}(t)\rangle$.

3.2.1. For Returns Drawn from a Binomial Distribution

In this section, we assume that the returns are drawn randomly from a binomial distribution $r(t) \in B\{-1,1\}$. Fig. 3.1 shows the distribution of the investor's budget for different time steps. The number of time steps needed to reach this stationary state depends on the initial conditions, the distribution of $r(t)$ and the additive constant a. For the conditions in Fig. 3.1, the number of time steps needed to reach a stationary distribution was of $t = 10^4$ time steps. One can clearly see that the stationary distribution is characterized by a fat tail described by a power-law distribution. This agrees with previous investigations [Levy and Solomon, 1996; Sornette, 1998a] which were discussed in Section 2.3.

We have determined the scaling exponent μ of the power law, Eq. (2.26), from the simulation data, leading to $\mu \approx 1$. And recalling Eq. (2.27), it can be shown that for binary returns $r(t) \in B\{-1,1\}$, the exponent μ is found by solving:

$$\sum_{\lambda=\{1-q,1+q\}} \lambda^{\mu}\, p(\lambda) = 1 \tag{3.1}$$

$$(1-q)^{\mu} + (1+q)^{\mu} = 2. \tag{3.2}$$

Note that for $q = 0.1$ and, in fact, for any value of $q \in [0,1]$, there are two possible exponent

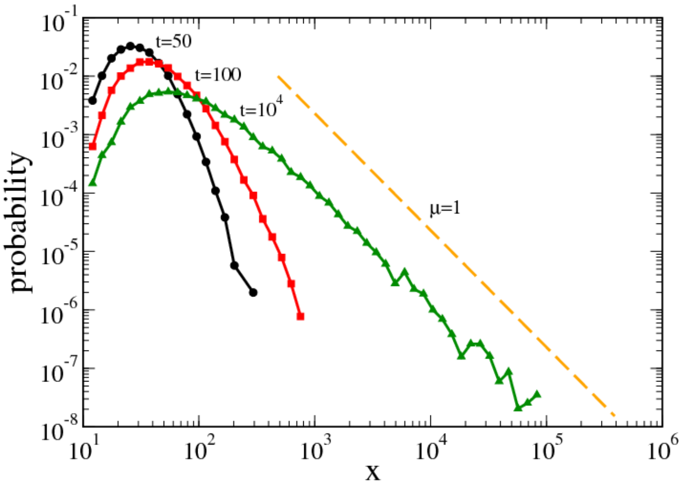


Figure 3.1.: For binary stochastic return distribution $r(t) \in B\{-1, 1\}$, proportion of investment $q(t) = 0.1$ and income $a = 0.5$: (bold) Investor's budget probability distribution $P(x, t)$ for different time steps. (dashed) Tail of stationary probability distribution $P_s(x)$, Eq. (2.26). The data is binned in logarithmic intervals of the same size. Further Parameters: initial budget $x(0) = 10$ and number of trials $N_a = 10^4$.

values: $\mu = 0$ or $\mu = 1$. This can be seen in Fig. 3.2, where the value of the left side in Eq. (3.1) is plotted for different exponents. Note that for small or large q, the two possible exponent values that satisfy Eq. 3.1 are the same. And from our simulation results, we notice that the power law in the tail of both distributions approaches $\mu \approx 1$.

Now we performed some simulations so that we could observe the evolution of the most probable budget value for different income and proportion of investment values. First, we start with a fixed proportion of investment and vary the income values. Fig. 3.3 (left) shows the evolution of the x_{mp} for a constant proportion of investment $q(t) = 0.1$ and different additive terms $a = \{0.1, 0.5, 1.0\}$. Notice that for simulations with small incomes, the x_{mp} reaches a stationary value faster than those simulations with larger income values. According to visual impression, for large income values, the value of x_{mp} reaches a stationary value after $t = 10^3$ time steps. Thus, it is clear from these simulations that for binomial RoI, the most probable budget value increases over the course of time, reaching a stationary state after a number of time steps. This results from the compensation between the tendency towards zero values (because of the multiplicative coefficient) and the tendency towards infinity positive values (because of the additive term). Now, by fixing the incomes at $a = 0.5$, we observe in Fig. 3.3 (right), the evolution of the x_{mp} for three different proportion of investment $q = \{0.1, 0.2, 0.5\}$. Note that for $q \geq 0.5$, the stationary values of x_{mp} are less than the initial budget of $x(0) = 10$. As expected, the agent increases his chances of receiving more profits on average if he decreases his proportion of investment because in this case, the process needs more time steps in order to reach a stationary distribution, which would allow the agent to accumulate more incomes during this time.

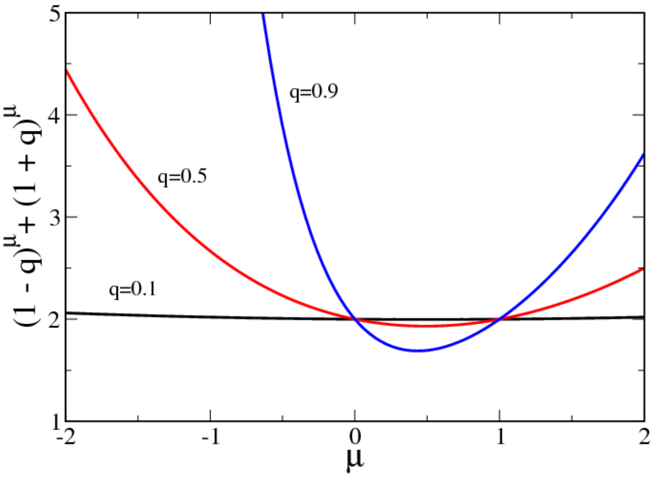


Figure 3.2.: Left side value in Eq. (3.1) for different exponent values, μ.

3.2.2. For Returns Drawn from a Uniform Distribution

In this section, we assume that the returns are drawn randomly from a uniform distribution, $r(t) \in U(-1, 1)$. First, we investigate the evolution of the budget probability distribution shown in Fig. 3.4 in a log-log plot. Note that as expected, the distribution shifts to larger positive values over the course of time, however, notice that the probability distribution does not change after a certain number of time steps; in other words, the process reaches a stationary probability distribution over the course of time. Observe that the tail of the distribution fits a power law Eq. (2.26) with $\mu \approx 1$. Recalling the multiplicative stochastic models with additive term discussed in Section 2.3.3, in this case, we have a multiplicative stochastic process like in Eq. (2.30) with a multiplicative coefficient $\lambda(t) = 1 + r(t)\, q$. For $a(t) = 0.1$, $q(t) = 0.1$ and $r(t) \sim U(-1, 1)$, this turns out to be a multiplicative coefficient of the form $\lambda(t) \sim U(0.9, 1.1)$. For these parameters, it can be shown that $\log \lambda = -0.001671$, which means the process has a pull towards zero. This, together with the repulsion from zero given by the additive term which is larger than zero, yields a distribution with a power law tail, Eq. (2.26). Furthermore, it can be shown that by solving Eq. (2.29) for λ, the exponent of the power law in the tail is $\mu \sim 1$.

As shown above, we performed some simulations to observe the evolution of the x_{mp} for different income and proportion of investment values. Fig. 3.5 (left) shows the evolution of the $\langle x_{mp} \rangle$ for a constant proportion of investment $q(t) = 0.1$ and different additive terms $a = \{0.1, 0.5, 0.9\}$. In the same manner as for binomial returns, the x_{mp} increases over the course of time until a stationary value is reached. However, according to visual impression, in this case, it takes more time steps to reach a stationary value. Note that the values for x_{mp} reached are much larger than those gathered for binomial RoI, and interestingly, for uniform RoI, the standard deviation increases more than for binomial RoI, maybe because for uniform RoI, the range of possible returns is larger and leads to more fluctuations on average. A visual impression indicates that for large income values, the value of $\langle x_{mp} \rangle$ reaches a stationary value after $t = 10^3$ time steps.

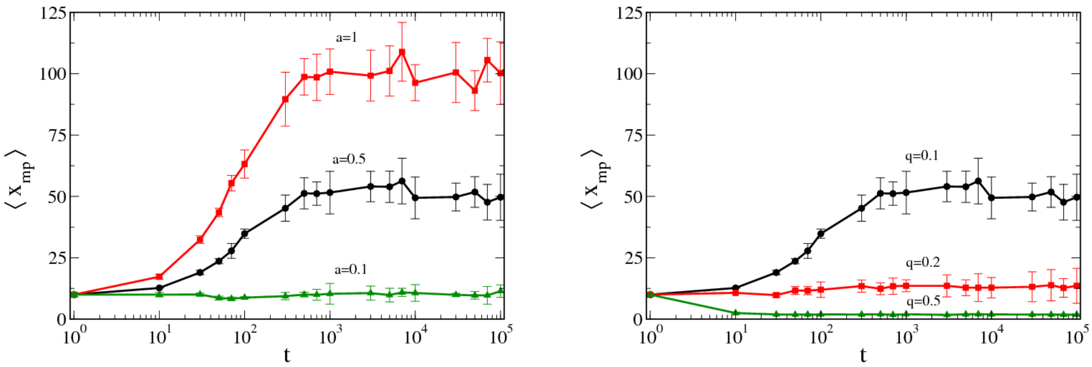


Figure 3.3.: Most probable budget value over the course of time, for binomial RoI, $r(t) \in B\{-1, 1\}$. For (left) constant proportion of investment $q(t) = 0.1$ and different income values a, and (right) constant income $a = 0.5$ and different constant proportion of investment values q. Further parameters: $N_a = 10^5$, $N_s = 10$ and $x(0) = 10$.

Now, assuming fixed incomes of $a = 0.5$, we observe in Fig. 3.5 (right) the evolution of the $\langle x_{mp} \rangle$ for three different proportion of investment values $q = \{0.1, 0.2, 0.5\}$. Note again that for $q \geq 0.5$, the stationary values of $\langle x_{mp} \rangle$ are less than the initial budget of $x(0) = 10$. For reference purposes we add in Fig. 3.5 the curve for $q = 0$, which means that the agent only accumulates income. In other words, the agent does not receive a return, note that the budget increases much more than when the agent takes a constant proportion of investment. It seems that it is much better for the agent to avoid investing in this kind of scenarios. This is true when the agent is dealing with random RoI; however, in Chapter 7 we deal with RoIs with periodicity where an agent forecasting the next return may have larger profits than a simpler one that only accumulates income.

Now, before continuing our analysis for returns drawn from a Gaussian distribution, we would first like to discuss what happen when returns are drawn from different ranges of values independently of the type of distribution. Firstly, note that if we set an unsymmetrical range of values giving more weight to negative or positive values, this leads of course to bankruptcy or richness respectively. Secondly, note that if the range of possible values is kept symmetrical and the range is increased or decreased, this is the same as modifying the proportion of investment value. This occurs because, as discussed previously, the value of λ in Eq. (2.30) corresponds in the investment model to $\lambda = 1 + q(t)\, r(t)$, and if the proportion of investment is decreased, this means that the range of values of $r(t)$ is also decreased. For example, in our previous computer experiment, we assumed $q = 0.1$ and $r(t) \sim U(-1, 1)$, which means $\lambda \sim U(0.9, 1.1)$. If the range of values for $r(t)$ changes to $r(t) \sim U(-0.5, 0.5)$, this would mean a $\lambda \sim U(0.95, 1.05)$. This previous range of values for λ can be also achieved for $r(t) \sim U(-1, 1)$ and a proportion of investment of $q = 0.05$. Thus, it can be seen that for an proportion of investment $q = 0.1$ and returns r drawn from $U(-0.1, 0.1)$, $U(-2, 2)$ and $U(-5, 5)$, the same range of values for λ can be obtained if the returns are now drawn from $U(-1, 1)$ and for the proportions of investment $q = 0.01$, $q = 0.2$ and $q = 0.5$ respectively. The evolution of the $\langle x_{mp} \rangle$ over the course of time for these different returns is shown in Fig. 3.6.

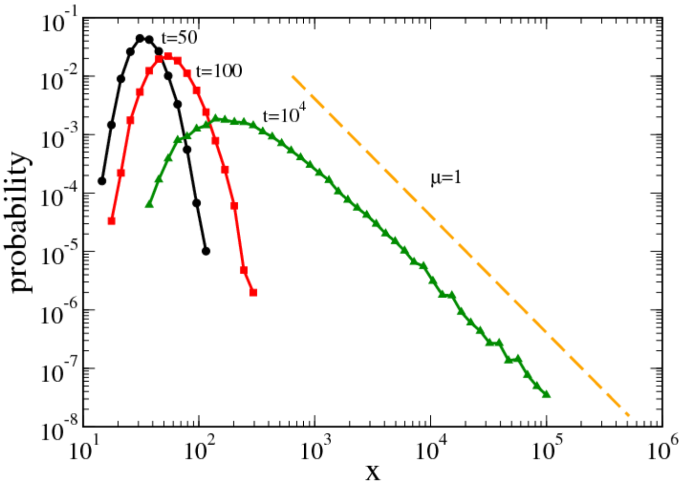


Figure 3.4.: For uniform stochastic return distributions $r(t) = U(-1, 1)$, assuming $q(t) = 0.1$ and $a = 0.5$: (bold) Investor's budget probability distribution $P(x, t)$ for different time steps. (dashed) Tail of stationary probability distribution $P_s(x)$, Eq. (2.26). The probabilities were estimated as in Fig. 3.1. Further parameters as in Fig. 3.1.

Fig. 3.6 also shows that the number of time steps needed for the process to reach a stationary state increase if either the range of values or the proportion of investment decrease. This occurs because of the weak influences of the random returns leading to more time steps until the large accumulation of income (pulling the process to ∞) is compensated in the long run by the investments in random returns (pulling the system to zero). Moreover, it can be seen that the slope of the increasing value of x_{mp} fits a power law, this is more clear for $r(t) \sim U(-0.1, 0.1)$, where the power law increasing x_{mp} goes from time step $t = 30$ to $t = 10^4$. Note that it would be interesting to find an analytical description of the number of time steps needed to reach a stationary x_{mp}, however, this is left as further work.

3.2.3. For Returns Drawn from Gaussian Distributions

We assume now that returns are drawn randomly from a Gaussian distribution $N(\mu, \sigma)$ with parameters: mean μ and standard deviation σ. Fig. 3.7 (left) shows the evolution of the probability distribution of the budget for different time steps. In this case we assume Gaussian returns $r(t) \sim N(\mu, \sigma)$, where $\mu = 0$ and $\sigma = 0.1$, and we observe that the probability also reaches a stationary distribution after some time steps and presents a power law distribution on the tail. However, if we compare this evolution of distributions with those with binomial or uniform returns, we can see that for Gaussian returns it takes more time steps. This can be expected because for $\sigma = 0.1$, the budget does not change as much over the course of time as it does for returns drawn from uniform, $U(-1, 1)$, or binomial, $B(-1, 1)$, distributions. This also means that a larger number of time steps would be needed to find a stationary distribution with a power law in the tail. To gain more insight into the number of time steps needed for the process to reach a stationary distribution, we

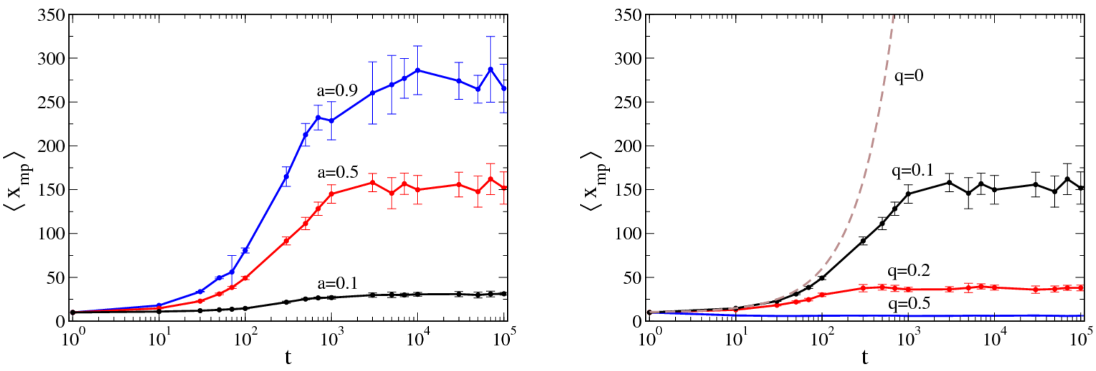


Figure 3.5.: Most probable budget value over time for uniform RoI, $r(t) \in U(-1, 1)$. (left) Constant proportion of investment $q(t) = 0.1$ and different income values a. (right) Constant income $a = 0.5$ and different constant proportion of investment values q. Further Parameters as in Fig. 3.3.

investigated the evolution of the budget distribution. Fig. 3.7 (right) shows the evolution of the probability distribution of the budget for $q = 0.1$ and $a = 0.1$ for different time steps and the same return's distribution $r(t) \sim N(0, 0.1)$. Note that we restricted the value of the returns to be within the range $r \in (-1, 1)$ in order to avoid unrealistic losses ($r < -1$, which would mean having debts) and realistic (but undesired for our study) over-gains, because of the violation of the pull and repulsion from zero in the dynamics of multiplicative stochastic processes with an additive term discussed previously in Section 2.3.3.

It is clear that in order to obtain the most probable value, we cannot limit our simulation to only $t = 10^4$ time steps because the process may not have reached a stationary distribution yet. The latter problem can be solved by fixing a larger number of time steps for the simulations, for example $t = 10^5$, however, in order to save computing time, we instead decided to determine the number of time steps t needed for stationary distributions, using the following statistical procedures: for the means and variance of the x_{mp}, *t-test* and *F-test* respectively, and for the evolution of the distribution the χ^2*-test*. Algorithm 2 shows the procedure used to numerically calculate the average most probable budget value $\langle x_{mp} \rangle$ for constant income a and constant proportion of investment q values in more detail. For further details regarding hypothesis testing strategies, refer to [Press et al., 1992] (pages 615-622) and [Cohen, 1998] (pages 117-130).

Thus, following Algorithm 2, Fig. 3.8 shows the evolution of the $\langle x_{mp} \rangle$ over time for Gaussian returns for: fixed proportion of investment $q(t) = 0.1$ and different a (left), and fixed income $a = 0.5$ and different q (right). The simulations were performed until no significant statistical differences were found between the average and variance of x_{mp} and the budget distributions. It can be seen that for larger incomes and smaller proportions of investment, a larger number of time steps are needed to reach both stationary distributions and stationary average values than for smaller incomes and larger investments. Interestingly, the most probable values are larger than those obtained for returns drawn randomly from binomial or uniform distributions. The reason for this is that the Gaussian returns lead to smaller RoI fluctuations, this means less win/loss due to random returns, so the agent's wealth is due more to the accumulation of incomes.

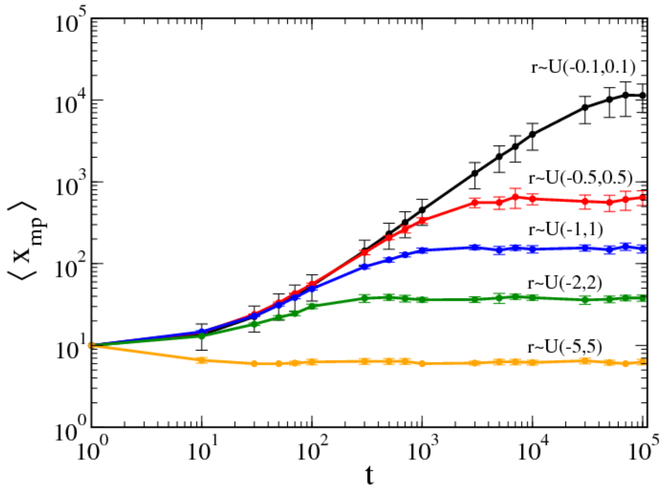


Figure 3.6.: Evolution of the most probable budget value over the course of time. Returns drawn randomly from uniform distributions with different range of values. Further parameters: $N_a = 10^5$, $N_s = 10$, $x(0) = 10$, $q(t) = 0.1$ and $a = 0.5$.

3.2.4. For Returns Using ARCH/GARCH Process

So far, we have analyzed the evolution of the budget for returns on investment (RoI) drawn randomly from binomial, uniform and Gaussian probability distributions. However, these type of RoIs are not typically found in investment instruments in real life, so in order to mirror reality more closely, we now consider another kind of RoI, which presents some correlations over time like those that can be seen in financial time series.

The *ARCH (Auto-Regressive Conditionally Heteroskedastic)* process, first introduced by Engle [1982], is a process that adds correlations over time. The simplest version of this process is an ARCH(1) process, which is described as follows:

$$r(t) = \epsilon(t)\sigma(t) \tag{3.5}$$

where $\epsilon(t) \sim i.i.d.N(0,1)$ and the conditional variance, $\sigma(t)$, is defined by,

$$\sigma(t)^2 = \alpha_0 + \alpha_1\, r(t-1)^2 \tag{3.6}$$

where $\alpha_0 > 0$ and $\alpha_1 \geq 0$

Another process used to generate artificial RoIs is the GARCH process [Bollerslev, 1986], which is a generalization of the ARCH process. A *GARCH (Generalized Auto-Regressive Conditionally Heteroskedastic)* process is a stochastic process with an autoregressive representation of the conditional variance and a moving average part. These processes present both distributions with fat tails and volatility clustering, two properties that are usually observed in financial time series [Bera and Higgings, 1993; Mantegna and Stanley, 2000]. In these investigations, a GARCH(1,1) process is considered, which generates a random return $r(t)$ based on the previous return $r(t-1)$ and the previous conditional variance value

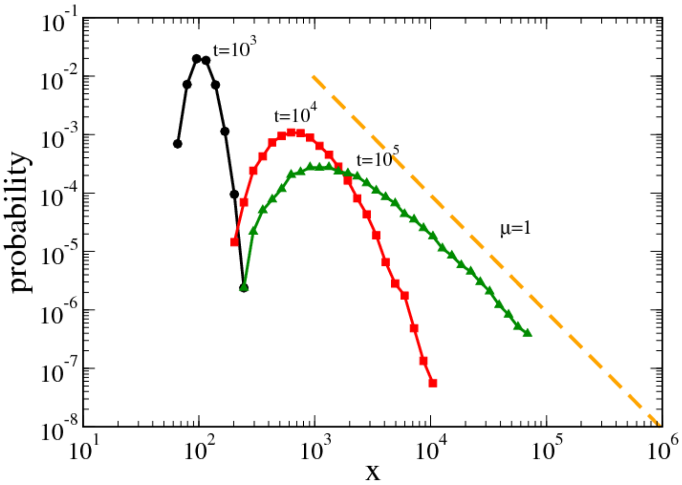


Figure 3.7.: Evolution of budget probability distribution over time for proportion of investment $q = 0.1$ and income $a = 0.1$. Returns drawn from a Gaussian stochastic process with $\mu = 0$ and $\sigma = 0.1$. Further parameters as in Fig. 3.1

$\sigma(t-1)$. Thus, the return $r(t)$ is calculated at every time step t as follows:

$$r(t) = \epsilon(t)\sigma(t), \tag{3.7}$$

where $\epsilon(t) \sim i.i.d.N(0,1)$ and the conditional variance $\sigma(t)$ is defined by

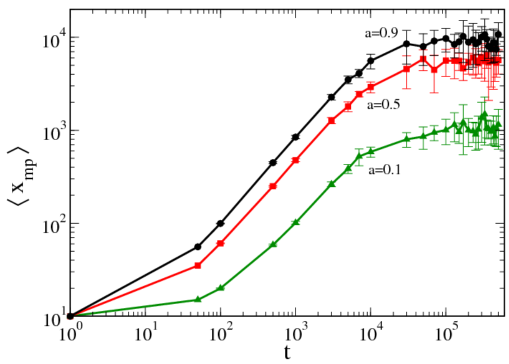
$$\sigma(t)^2 = \alpha_0 + \alpha_1\, r(t-1)^2 + \beta_1\, \sigma(t-1)^2, \tag{3.8}$$

where $\alpha_0 > 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$. For more details on Auto-Regressive Conditionally Heteroskedastic processes, refer to Appendix 10.2.

Fig. 3.9 shows the realizations of ARCH(1) and GARCH(1,1) processes. Note that in both processes there are some time steps in which there are large fluctuations in the returns and others with relatively small changes; this property of the time series is called volatility clustering. Note that when using these processes, the return may exceed the range $[-1, 1]$, as it was defined in Section 2.2. These values are not being considered in the dynamics of the model, i.e. we have to truncate the process when simulating the investment dynamic in Eq. (2.9).

Fig. 3.10 offers yet more insight into the dynamics of the investment model for different type of return by showing the evolution of the budget probability distribution over time for ARCH(1) and GARCH(1,1) processes. It can be seen that a stationary distribution was reached after approximately $t = 10^4$ time steps. However, as it was done for Gaussian returns, when performing simulations for the ARCH/GARCH processes, the value of the returns was restricted to the range $r \in (-1, 1)$. This was done in order to avoid unrealistic losses and realistic over-gains which are undesirable in our study because they violate the properties of pull and repulsion from zero for multiplicative processes with power law distributions discussed in Section 2.3.3.

Now, using Algorithm 2 (Section 3.2.3), some simulations were performed in order to



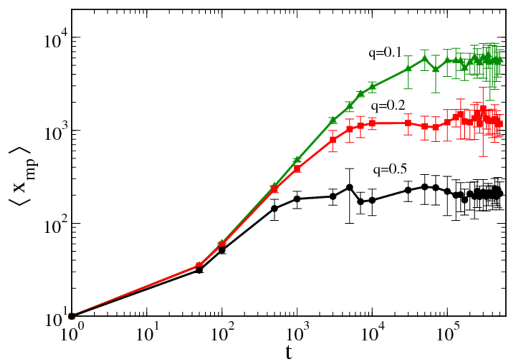


Figure 3.8.: Most probable budget value over time, for Gaussian RoI, $r(t) \in N(0, 0.1)$. (left) Constant proportion of investment $q(t) = 0.1$ and different income values of a. (right) Constant income $a = 0.5$ and different constant proportion of investment values of q. For $\tau = 20000$ sampling time steps and $p_v = 0.1$. Further Parameters as in Fig. 3.3

observe the evolution of the most probable budget value for different income and proportion of investment values. Fig. 3.11 shows the evolution of the x_{mp} over time for ARCH(1) returns as in Fig. 3.9 (left), for different constant proportions of investment and different additive terms. Visual impression would suggest that $\langle x_{mp} \rangle$ reaches a stationary value faster than for Gaussian RoI, but if we compare this with binomial or uniform RoI, a larger number of time steps are needed to reach stationary values. Obviously, this depends on the parameters of the processes and the range of values for the distribution. Note also that for these particular parameter values, the $\langle x_{mp} \rangle$ obtained for ARCH or GARCH returns is much larger than the $\langle x_{mp} \rangle$ gathered for binomial and uniform RoI. This is may be due to the property of volatility clustering which yields from time to time small and large fluctuations in the returns.

3.3. Analytic Solution

In Section 2.2, we started analyzing the dynamics of the investment model using basic analytical tools, showing in Eq. (2.13) a solution for non-random returns and in Eq. (2.14) a description of the time-dependent process for random returns. Note that there may be different ways to treat the stochastic dynamics of Eq. (2.9) [Haan and Karandikar, 1989]. In the following, an analysis of the dynamics using the $\mathcal{Z}$-transform [Jury, 1973] is presented and afterwards the analytical derivation of the stationary probability distribution of the budget is obtained.

3.3.1. Solution Using Z-transform

For simplicity, we start the analytical analysis by treating the stochastic dynamics of Eq. (2.30) directly [Navarro-Barrientos et al., 2008b]. For this case, assuming in Eq. (2.9) $a(t) = a$, the following formal solution using the $\mathcal{Z}$-transform [Jury, 1973] can be obtained.

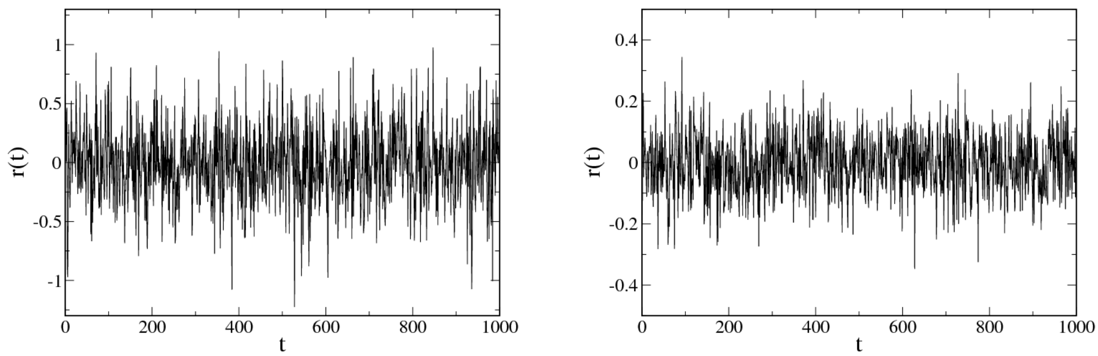


Figure 3.9.: Stochastic processes with correlation over time: (left) ARCH(1) for $\alpha_0 = \alpha_1 = 0.1$ and (right) GARCH(1,1) for $\alpha_0 = \alpha_1 = \beta_1 = 0.01$.

Rewriting Eq. (2.30) as

$$a \;=\; x(t+1) - \lambda x(t) \tag{3.9}$$

the $\mathcal{Z}$-transform leads to

$$a \sum_{n=0}^{\infty} \frac{1}{z^n} = a\,\frac{z}{z-1} = z\,[X(z) - x(0)] - \lambda X(z) \tag{3.10}$$

where $X(z)$ is given by

$$X(z) = x(0)\,\frac{z}{z-\lambda} + a\,\frac{z}{(z-1)(z-\lambda)} \tag{3.11}$$

Now, using the inverse transform

$$X(z^{-1}) = \left[x(0) - \frac{a}{1-\lambda}\right]\,\frac{1}{1-\lambda z} + \frac{a}{1-\lambda}\,\frac{1}{1-z} \tag{3.12}$$

the solution for $x(t)$ can be found as

$$\begin{aligned} x(t) &= \frac{1}{t!}\,\partial_z^t\, X\left(\frac{1}{z}\right) && (3.13)\\ &= \lambda^t\,\left[x(0) - \frac{a}{1-\lambda}\right] + \frac{a}{1-\lambda} && (3.14)\\ &= \lambda^t\,x(0) + a\,\frac{1-\lambda^t}{1-\lambda} && (3.15)\\ &= \lambda^t\,x(0) + a\,\sum_{s=0}^{t-1}\lambda^s && (3.16) \end{aligned}$$

From this solution, we see that the decisive condition of λ for a well-defined solution reads:

$$|\lambda| < 1 \tag{3.17}$$

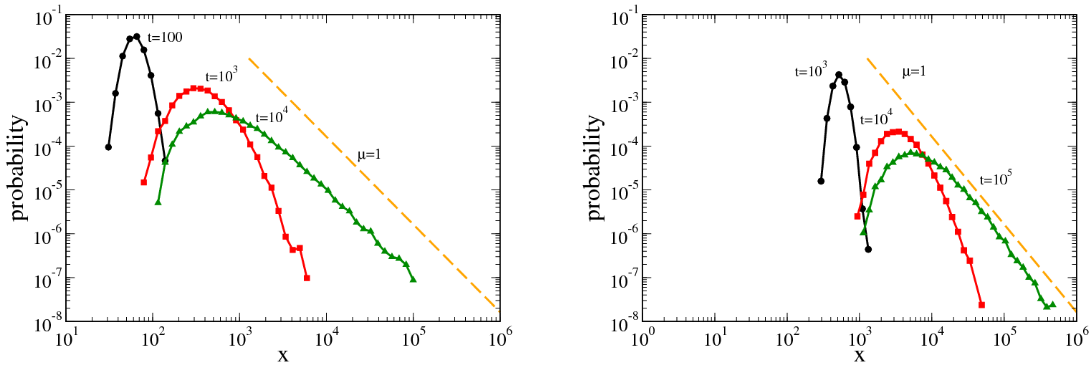


Figure 3.10.: Evolution of budget probability distribution over time for proportion of investment $q = 0.1$, and $a = 0.5$. Returns obtained from: (left) an ARCH(1) process with $\alpha_0 = \alpha_1 = 0.1$ (see Fig. 3.9 (left)); and (right) a GARCH(1,1) process with $\alpha_0 = \alpha_1 = \beta_1 = 0.01$ (see Fig. 3.9 (right)). Further parameters as in Fig. 3.1.

which, recalling Section 2.3.3, agrees with the determination of $\langle \log |\lambda| \rangle < 0$ by Kesten [1973], which was obtained from the treatment of the probability distribution $P(x)$.

3.3.2. Stationary Probability Distribution

As we noted in Section 2.3.3, different researchers have analyzed the probability distribution of multiplicative additive processes of the form in Eq. (2.30). An interesting approach is the one taken by Malcai et al. [2002], who take into account additional couplings between these processes, extending the model towards a generalized Lotka-Volterra model of the form:

$$x_i(t+1) = x_i(t)\ [1 + \lambda(t)] + a\bar{x}(t) - Cx_i(t). \tag{3.18}$$

Note that because of the additional couplings between different individual stochastic processes, $x_i(t)$, the dynamics of this model are much complex than the dynamics for the investment model proposed in this thesis, Eq. (2.9). Instead of a small, but constant income a the term $a\bar{x}(t)$ considers a *global coupling* via the mean budget $\bar{x}(t)$ of all agents (which may be related to general publicly funded services). Moreover, the third term $Cx_i(t)$ describes direct interactions between different agents, as C is a function dependent on other x_j. These interactions may account for competition for limited resources and saturation effects in the dynamics. Even if, after some approximations discussed by Malcai et al. [2002], these additional influences are small, they may still affect the general solutions for the underlying probability distributions. Moreover, to solve the dynamics of Eq. (3.18), Richmond [2001] proposed a general framework of multiplicative processes of the form:

$$\Delta x(t) = \eta(t) G[x(t)] + F[x(t)] \tag{3.19}$$

where $\eta(t)$ is a stochastic variable. The author finds a steady state solution for Eq. (3.19) of the form:

$$P(x) = \frac{1}{Z} \exp\left[-\psi(x) - \log G(x)\right] \tag{3.20}$$

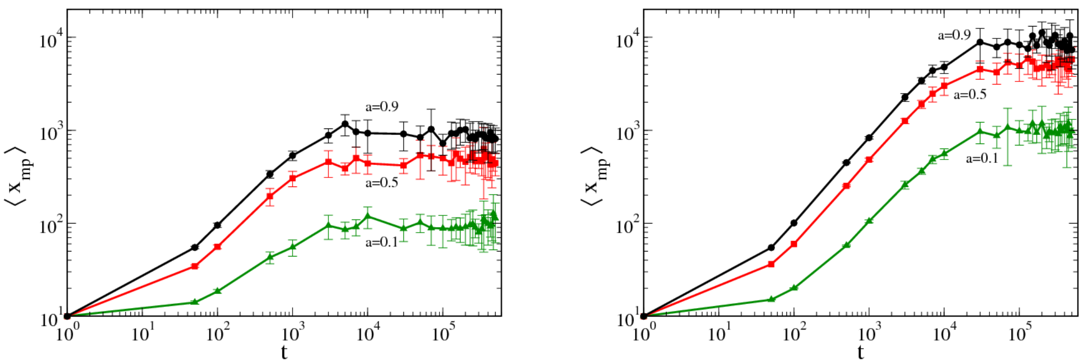


Figure 3.11.: Most probable budget value over time, for RoI obtained from the processes: (left) ARCH(1) with $\alpha_0 = \alpha_1 = \beta_1 = 0.1$; and (right) GARCH(1,1) with $\alpha_0 = \alpha_1 = \beta_1 = 0.01$, both for a constant proportion of investment $q(t) = 0.1$ and different income values a. Further parameters see Fig. 3.3

where:

$$\psi(x) = -\frac{1}{D}\int^{x} F/(G)^2 \mathrm{d}x' \tag{3.21}$$

and Z is a normalization factor.

A different approach is to use the Fokker-Planck analytical approximation proposed by Sornette and Cont [1997] for multiplicative additive random processes, Eq. (2.39. Navarro-Barrientos et al. [2008b] shows a stationary solution for the investment model in Eq. (2.9) based on the approximation of Sornette and Cont [1997], in the following, this derivation is presented. For $a(t) = a$ in Eq. (2.9) and the notation for the diffusion constant

$$D_x = \langle x^2 \rangle - \langle x \rangle^2, \tag{3.22}$$

we find that in the stationary case:

$$\begin{aligned} 0 &= a\, e^{-w} P_s(w) - \left(\langle \log \lambda \rangle + a\, e^{-w}\right) \partial_w P_s(w) \\ &+ \frac{D_{\log \lambda}}{2} \partial_w^2 P_s(w) \\ &= -\partial_w \left[\left(\langle \log \lambda \rangle + a\, e^{-w}\right) P_s(w) - \frac{D_{\log \lambda}}{2} \partial_w P_s(w) \right]. \end{aligned}$$

This can be solved for a stationary solution as follows:

$$
\begin{aligned}
0 &= (\langle \log \lambda \rangle + a\, e^{-w})\, P_s(w) - \frac{D_{\log \lambda}}{2} \partial_w P_s(w) \\
\partial_w \log P_s(w) &= \frac{2\,\langle \log \lambda \rangle + 2\,a\, e^{-w}}{D_{\log \lambda}} \\
\log P_s(w) &= \log \mathcal{N} + \frac{2\,\langle \log \lambda \rangle\, w - 2\,a\, e^{-w}}{D_{\log \lambda}} \\
P_s(w) &= \mathcal{N} \exp\left(\frac{2\,\langle \log \lambda \rangle\, w - 2\,a\, e^{-w}}{D_{\log \lambda}} \right)
\end{aligned} \tag{3.23}
$$

with normalization $\mathcal{N}$.

The corresponding stationary probability distribution for $w = \log x$ is then recovered by the chain rule as follows:

$$
\begin{aligned}
P_s(x) &= P_s(w(x))\, \frac{dw}{dx} \\
&= \frac{\mathcal{N}}{x} \exp\left(\frac{\langle \log \lambda \rangle\ \log x - \frac{2\,a}{x}}{D_{\log \lambda}} \right) \\
&= \mathcal{N}\, x^{\frac{2\,\langle \log \lambda \rangle}{D_{\log \lambda}} - 1} \exp\left(-\frac{2\,a}{D_{\log \lambda}\, x} \right)
\end{aligned} \tag{3.24}
$$

and the normalization can be calculated:

$$
\begin{aligned}
1 \stackrel{!}{=} \mathcal{N} &\int_0^\infty x^{\frac{2\,\langle \log \lambda \rangle}{D_{\log \lambda}} - 1} \exp\left(-\frac{2\,a}{D_{\log \lambda}\, x} \right) dx \\
= \mathcal{N} &\int_0^\infty y^{-\left(1 + \frac{2\,\langle \log \lambda \rangle}{D_{\log \lambda}}\right)} \exp\left(-\frac{2\,a\,y}{D_{\log \lambda}} \right) dx \\
= \mathcal{N} &\left(\frac{D_{\log \lambda}}{2\,a} \right)^{-\frac{2\,\langle \log \lambda \rangle}{D_{\log \lambda}}} \Gamma\left(-\frac{2\,\langle \log \lambda \rangle}{D_{\log \lambda}} \right).
\end{aligned}
$$

For simplicity, if we redefine

$$
\mu \ := \ -\frac{2\,\langle \log \lambda \rangle}{D_{\log \lambda}}, \tag{3.25}
$$

the stationary probability distribution of the budget x can be described by [Navarro-Barrientos et al., 2008b]:

$$
P_s(x) = \frac{\left(\frac{2\,a}{D_{\log \lambda}} \right)^{\mu}}{\Gamma(\mu)}\, x^{-(1+\mu)} \exp\left(-\frac{2\,a}{D_{\log \lambda}\, x} \right). \tag{3.26}
$$

Note, that the importance of $\langle \log \lambda \rangle < 0$ is mirrored here by the fact that the Gamma function diverges for $\mu \to 0$. Also note that for large x, Eq. (3.26) can be reduced to the power-law distribution, Eq. (2.26).

We have shown in Section 3.2.1 and in Section 3.2.2 for binomial and uniform distributions respectively, that the evolution of the budget distribution in our computer experiments reaches a stationary distribution. Now, we compare the results from the computer simulations with the analytical solution given in Eq. (3.26). Figures 3.12 show that the stationary distribution is characterized by a fat tail described by a power-law distribution. This agrees with previous investigations [Levy and Solomon, 1996; Sornette, 1998a] and was already discussed in Section 2.3.3. We have determined the scaling exponent μ of the power law in Eq. (2.26) from the simulation data for different stochastic processes and it can be shown that they agree with our analytical investigations.

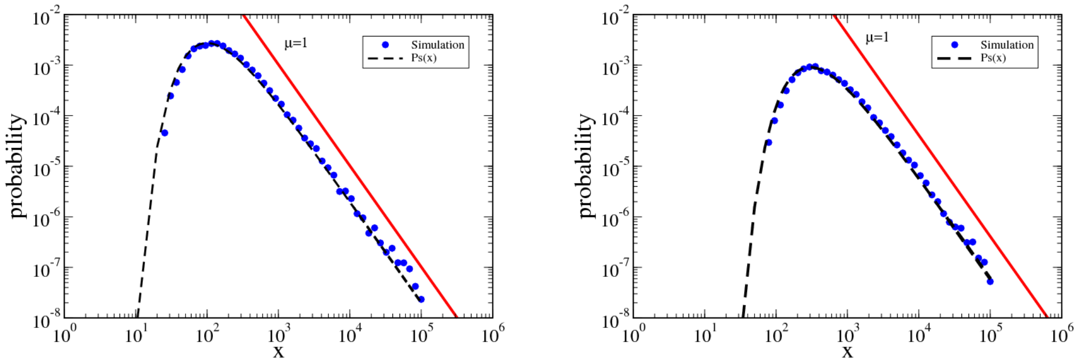


Figure 3.12.: Budget stationary probability distribution $P_s(x)$ (estimated from frequencies after $t = 10^4$): (left) Binary stochastic return distribution $r(t) = B\{-1, 1\}$. (right) Uniform stochastic return distributions $r(t) = U(-1, 1)$. In both cases, $x(0) = 10$, $q(t) = 0.1$ and $a = 1$. Data is binned in logarithmic intervals of the same size. The dashed lines show the theoretical prediction of these curves by Eq. (3.26).

Finally, note that Eq. (3.26) is a special form of the general solution

$$P_s(x) = \frac{1}{G^2(x)} \exp\left(\frac{2}{D}\int^x \frac{F(x')}{G^2(x')}dx'\right) \tag{3.27}$$

obtained by Malcai et al. [2002]; Richmond [2001]. If we use $F(x) = a$ and $G(x) = x$ in accordance with the stochastic process defined in Eq. (2.9), this would lead to the (non-normalized) solution $P_s(x) = x^{-2}\exp(-2a/Dx)$. This is in agreement with Eq. (3.26) because of $\mu \approx 1$ in our case. The solution discussed by Malcai et al. [2002] is, however, different from ours because the authors consider $F(x) = a(1-x)$ and $G(x) = x$, which eventually lead to the stationary distribution $P_s(x) = x^{-2(1-a/D)}\exp(-2a/Dx)$ and consequently to $\mu = 1 + (2a/D)$. Note that this is true only in the limit of $a \ll D$ which holds for the case discussed by Malcai et al. [2002] because of e.g. $a \approx D/4$, however, this is hardly satisfied in our model. Apart from these differences, the emergence of the stable power-law distribution in Eq. (2.26) for large x has been discussed in [Blank and Solomon, 2000; Levy and Solomon, 1996; Richmond, 2001; Richmond and Solomon, 2001; Solomon and Richmond, 2001a,b, 2002; Sornette, 1998a; Sornette and Cont, 1997].

In the following chapter, the properties of $\log\lambda$ are shown more in detail, which helps us to understand the role that q plays in the evolution of $\log\lambda$. The number of time steps needed to reach stationary distributions is also discussed and it is shown that a scaling function for the most probable budget value can be found for a constant proportion of investment, fixed income and some properties of the returns.

3.4. Summary and Extensions

In this chapter we learned the following:

If the agent decides to invest a fixed ratio of its budget, the distribution of the budget converges over the course of time to a stationary distribution with a power law in the tail. It is shown that the number of time steps needed for the process to converge may vary depending on the type of returns. This chapter also presented a useful approach for ensuring stationary distributions and stationary most probable budget x_{mp} values based on hypothesis testing methods.

Further work:

It would be interesting to find an analytical expression for the evolution of the probability distribution of the budget $p(x, t)$. This expression would also be useful for returns that do not pull the process toward zero values and is especially useful for returns in the range $r(t) \in (-\infty, \infty)$. It would be also interesting to analyze the dynamics of constant proportions of investment and incomes for other types of returns, like those that can be drawn from geometric Brownian motion. Finally, it would be useful to find an analytical expression to determine the number of time steps needed for the distribution of the budget to reach a stationary state.

Algorithm 2: Calculation of the average most probable budget value $\langle x_{mp} \rangle$

Input: Initial budget $x(0)$, proportion of investment q, income a, return on ivnestment $r(t)$, maximum number of time steps $t_{\max}$, number of time steps for sampling τ, number of trials N_a, number of distributions N_s and pv the desired *p value* for the significance tests.

Output: Evolution of the $\langle x_{mp} \rangle$ for a number of time steps t with sampling every τ time steps.

Initialize N_s groups of simulations where each group is used to simulate $N_a = 10^4$ trials of the dynamic in Eq. (2.9).

Initialize the number of time steps $t = 0$

for $t < t_{\max}$ **do**

  Run the $N_s \times N_a$ trials for time step $t = t + 1$

  if $t \mod \tau \equiv 0$ **then**

    Obtain the budget distribution for each group of simulations, leading to N_s distributions.

    Obtain the x_{mp} of the N_s distributions and calculate and save $\langle x_{mp} \rangle$ and $Var(x_{mp})$.

    Using the *Student's t-test*, measure the significance of the difference between the mean at the current time step t with the mean at time step $t - \tau$, where the null and alternative hypothesis are respectively:

$$H_0 : \langle x_{mp}(t) \rangle = \langle x_{mp}(t - \tau) \rangle \tag{3.3}$$

$$H_1 : \langle x_{mp}(t) \rangle \neq \langle x_{mp}(t - \tau) \rangle \,. \tag{3.4}$$

    Calculate the p value pv_{means}.

    Using the *F-test*, test for equal variances of the most probable values, $Var(x_{mp}(t')) = Var(x_{mp}(t' - \tau))$ and calculate the p value pv_{vars}.

    Using the χ^2*-test* test for equal distributions of budgets $p(x, t') = p(x, t' - \tau)$. and calculate the p value pv_{dists}

    if *the p values:* pv_{means}*,* pv_{vars} *and* pv_{dists} *are all larger than* pv **then**

      The null hypothesis cannot be rejected, store the evolution of the $\langle x_{mp} \rangle$ and $Var(x_{mp})$ for every τ steps and exit loop.

    end

  end

end

4. Scaling of the Most Probable Budget Value

This chapter investigates more in detail the influence of the stochastic factors on the evolution of the budget and presents a scaling function for the most probable budget value.

4.1. Introduction

In the previous chapter, we investigated the influence of different fixed investment proportions and fixed incomes on the budget distribution and its evolution over time. In this chapter, in Section 4.2 the mean value of the profits resulting from the product between the returns and the proportion of investment are investigated for different types of returns (multiplicative coefficient in Equation (2.9)). Thus, in Section 4.3, using simulations, the influence of the multiplicative coefficient in Equation (2.9) on the stationary most probable budget value x_{mp} is investigated and a scaling function for x_{mp} is presented for the value of the proportion of investment q, the income a and some properties of the returns. Finally, in Section 4.4, the simulation results are corroborated by obtainig the scaling function analytically.

4.2. Analysis of the Mean Value of the Profits

In this section, we investigate the influence of the product of the returns and the proportion of investment, i.e $\lambda = 1 + q\, r(t)$ in Equation (2.30), in the profits of the agent. We are particularly interested in those λ values that lead the process to a stationary state. This process is presented in the following analytical and numerical calculation of $\langle \log \lambda \rangle = \int_{\mathbb{R}} \log \lambda p(\lambda) d\lambda$ analytically and via simulations.

First, note that for binomial returns, $r_B = r \sim B\{-1; 1\}$, the value of $\langle \log \lambda \rangle$ can be calculated as follows:

$$\begin{aligned} \langle \log \lambda \rangle &= \frac{1}{2} \left[\log(1-q) + \log(1+q) \right] \\ &= \frac{1}{2} \log{(1-q)(1+q)} \\ &= \frac{1}{2} \log{(1-q^2)} \end{aligned} \tag{4.1}$$

For uniform returns, $r_U = r \sim U(-1,1)$, we have:

$$\begin{aligned}
\langle \log \lambda \rangle &= \frac{1}{2\,q} \int_{1-q}^{1+q} \log \lambda d\lambda \\
&= \frac{1}{2\,q} \left[(1+q) \log (1+q) - (1+q) - (1-q) \log (1-q) + (1-q) \right] \\
&= \frac{1}{2\,q} \left[\log (1+q) + q \log (1+q) - \log (1-q) + q \log (1-q) - 2\,q \right] \\
&= \frac{1}{2} \left[\frac{1}{q} \log \left(\frac{1+q}{1-q} \right) + \log \left(1 - q^2 \right) - 2 \right] \\
&= \frac{1}{2} \log \left(1 - q^2 \right) + \frac{1}{2\,q} \log \left(\frac{1+q}{1-q} \right) - 1 \qquad (4.2)
\end{aligned}$$

For these two types of returns, it can be seen that when $q \to 0$ then $\langle \log \lambda \rangle \to 0$, and when $q \to 1$ then $\langle \log \lambda \rangle \to -\infty$. Moreover, it can be shown that for $q > 0$, the profits for uniform returns are equal or larger than for binomial returns, i.e. $\langle \log(1 + r_B\, q) \rangle \leq \langle \log(1 - r_U\, q) \rangle$. This can be seen in Eq. (4.2), where the second term $\frac{1}{2\,q} \log \left(\frac{1+q}{1-q} \right) \geq 1$ for $q \in (0,1)$.

Now, for Gaussian returns, $r_N = r \sim N(0,\sigma)$, we note that if the expected value of the returns is zero, then a Taylor expansion up to the second order can be used, for this we have:

$$\begin{aligned}
\langle \log \lambda \rangle &= \int_{\mathbb{R}} \left(\lambda - 1 - \frac{(\lambda - 1)}{2} \right) p(\lambda) d\lambda \\
&= \frac{-q^2 \sigma^2}{2} \qquad (4.3)
\end{aligned}$$

For these type of returns, it is clear that as q increases, the value of $\langle \log(1 + r\, q) \rangle$ also increases. This is also tested by means of simulations. For this, the value of $\langle \log(1 + r\, q) \rangle$ was calculated numerically for different fixed values of q and for r a randomly drawn from the distributions: binomial $B(-1,1)$, uniform $U(-1,1)$ and Gaussian $N(0,0.1)$. Fig. 4.1 shows the value $100^{\langle \log \lambda \rangle}$ for different q and for different returns. Notice that each point in the plot corresponds to the average value calculated for 10^4 values of $\log \lambda$. The dashed lines show the analytical results for binomial, uniform and Gaussian returns using Eq. (4.1) and Eq. (4.2), and Eq. (4.3), respectively. For comparison, a change of scale in the *y-axis* from $\langle \log \lambda \rangle$ to $100^{\langle \log \lambda \rangle}$ was performed. It is clear that for all returns we consider in these experiments $\langle \log \lambda \rangle < 0$, and it can be seen that for binomial and uniform returns, it can be seen that as q increases the value of $\langle \log \lambda \rangle$ decreases much more than for Gaussian returns.

Moreover, notice that for binomial returns, there is an inflection point at approximately the same value in both y and x axes, i.e. for $q = 0.5$ it seems that $100^{\langle \log(1 + r_B\, q) \rangle} \approx q$. This interesting fact can be proved analytically, assuming that $g(q) = \langle \log(1 + r_B\, q) \rangle$ and $f(q) = A^{g(q)}$, with $A = 100$. A necessary condition for q to be an inflection point is to show that $f''(q) = 0$ as follows:

$$\begin{aligned}
f(q) &= A^{g(q)} \\
f'(q) &= Ag(q) \log Ag'(q) \\
f''(q) &= f'(q) \log Ag'(q) + f'(q) g''(q).
\end{aligned}$$

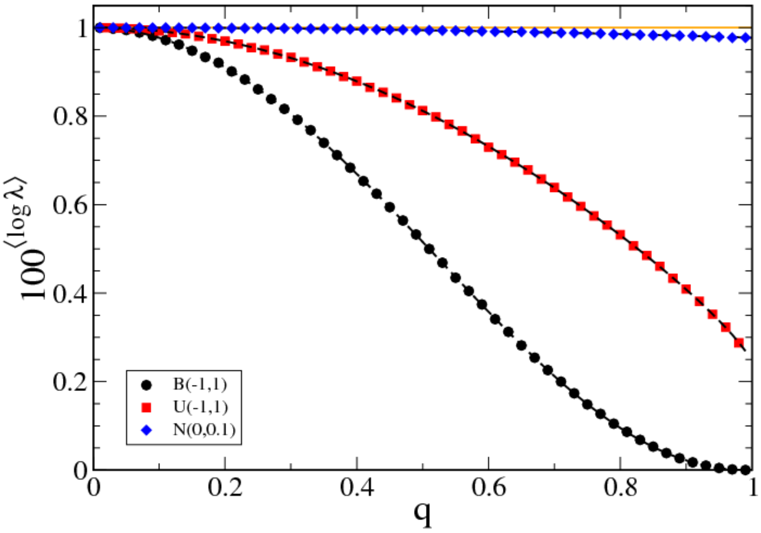


Figure 4.1.: Calculation of $\langle \log \lambda \rangle < 0$ via simulations (symbols) and analytically (dashed-lines), for returns drawn from binomial $B(-1, 1)$, uniform $U(-1, 1)$, and Gaussian $N(0, 0.1)$ distributions.

For the inflection point we have:

$$\begin{aligned}
0 &= f''(q) \\
0 &= A^{g(q)} \log^2 (A)(g'(q))^2 + A^{g(q)} \log (A) g''(q) \\
0 &= \log (A)(g'(q))^2 + g''(q).
\end{aligned}$$

Now, using Eq. 4.1 we have:

$$\begin{aligned}
g(q) &= \frac{1}{2} \log(1 - q^2) \\
g'(q) &= \frac{-q}{1 - q^2} \\
g''(q) &= \frac{-1 - q^2}{(1 - q^2)^2}.
\end{aligned}$$

Finally, the value of q at the inflection point can be found as follows:

$$\begin{aligned}
0 &= \log (A)(\frac{-q}{1 - q^2})^2 - \frac{(1 + q^2)}{(1 - q^2)^2} \\
0 &= \log (A) q^2 - q^2 - 1 \\
q &= \sqrt{\frac{1}{\log(A) - 1}};
\end{aligned}$$

for $A = 100$ this leads to $q \approx 0.526$. It is possible that this inflection point $q \approx 0.5$ is related to the fact that the returns can be $r = -1$ or $r = +1$, therefore showing a tendency to a much larger pull to zero if $q > 0.5$ and vice versa.

Finally, note that for Gaussian returns, the values of $\langle \log \lambda \rangle$ in Fig. 4.1 are nearer to zero values than for binomial and uniform returns. This is the reason for the larger number of time steps needed to reach a stationary value of $\langle x_{mp} \rangle$ for Gaussian returns in Fig. 3.8 with respect to the number of time steps needed for binomial returns in Fig. 3.3 and uniform returns in Fig. 3.5.

4.3. Simulation Experiments

In this section, the relationship between the agent's most probable budget x_{mp} is investigated for different constant proportion of investment q and different income values a. We start our discussion by presenting the results from a number of computer simulations performed to find a scaling function for x_{mp} given the parameters q, a and the RoI. Afterwards, in Section 4.4 the analytical solution that corroborates the simulation results is shown. Recalling Fig. 3.3 for binomial returns and Fig. 3.5 for uniform returns, note that these results suggest the presence of a scaling function between the most probable value x_{mp} and the parameters characterizing the stochastic dynamics Eq. (2.9). This scaling is investigated numerically following the same approach as in Section 3.2, where Algorithm 2 (Section 3.2.3) was used to obtain the $\langle x_{mp} \rangle$ for different constant values of q and a. For the scaling, the values of q and a were gradually increased for each simulation in order to cover the whole range of values. We assumed the following parameters for the simulations: initial budget $x(0) = 10$, number of trials for the agent's dynamics $N_a = 10^4$, number of trials for the simulation $N_s = 20$, and a pvalue for stationary distributions of $p_v = 0.1$.

4.3.1. For Binomial and Uniform Returns

First, we analyze the case where returns are drawn from a binomial distribution $B\{-1, 1\}$ and a uniform distribution $U(-1, 1)$. Fig. 4.2 shows the $\langle x_{\mathrm{mp}} \rangle$ for binomial and uniform returns. The results plotted against the variable a/q^2 clearly show a straight line, which allows for the scaling:

$$x_{\mathrm{mp}} = c \cdot \frac{a}{q^2}. \tag{4.4}$$

Note that this equation is of special interest for any agent investing in random environments as it describes the most probable budget value that can be obtained in the long-run given the constant income a, the constant proportion of investment q and the coefficient c, which depends on some properties of the returns. Thus, using an implementation of the nonlinear least-squares Levenberg-Marquardt algorithm, the resulting $\langle x_{\mathrm{mp}} \rangle$ values from the simulations were fitted to the scaling function Eq. 4.4, leading to the empirical coefficient values of $c = 0.961 \pm 0.006$ for binomial returns, $r(t) \sim B{-1, 1}$ and $c = 3.149 \pm 0.014$ for uniform returns $r(t) \sim U(-1, 1)$.

4.3.2. For Returns with Broader Distributions

If returns are drawn from a Gaussian distribution, note first that the Gaussian distribution is defined for $\mathbb{R}$, which means that returns may be less than -1 leading to debts and greater than 1 leading to an over-enrichment. If we consider only returns in the range $r(t) \in [-1, 1]$, then we need to take into account that the underlying stochastic process for $r(t)$ may frequently lead to values outside the interval $(-1, +1)$, which must then to be discarded. This is also the case for returns based on the ARCH/GARCH processes.

In order to deal with broader distributions for the RoI, there are two options:

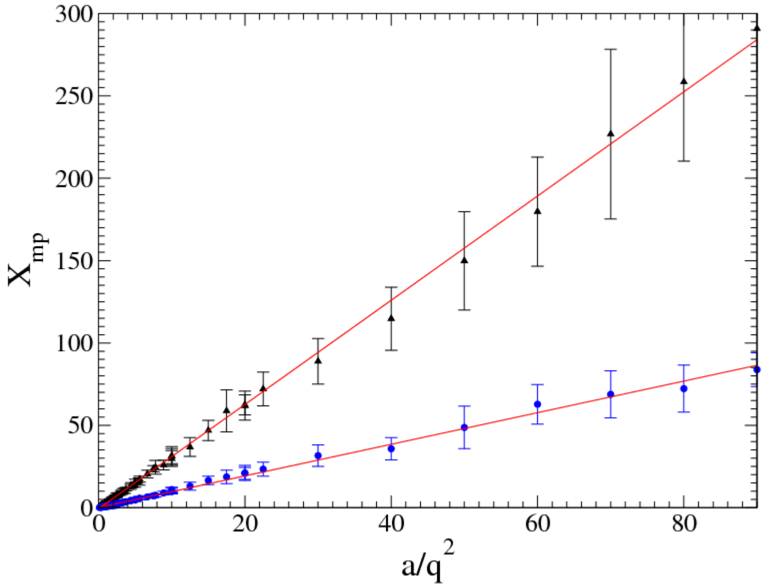


Figure 4.2.: Most probable budget value vs. scaled variable a/q^2 for the binary stochastic process $r(t) \in \{-1, 1\}$ (o) and for the uniform stochastic process $r(t) \in U(-1, 1)$ ($\triangle$). Each set was plotted by varying a and q over the range of $[0.1, 0.9]$ in 0.1 increments, giving a total of 81 data points per combination. The value of the slope found for the numerical simulations is $c = 0.961 \pm 0.006$ for the binary stochastic process and $c = 3.149 \pm 0.014$ for the uniform stochastic process. Further parameters: $x(0) = 10$, $N_a = 10^4$, $N_s = 20$, and $p_v = 0.1$.

(i) The "outliers" or returns outside the interval $[-1, 1]$ are taken into account. For positive returns, this only indicates an over-enrichment of the agent, however, for negative returns this results in a particular situation in which the agent is either bankrupt or owes money. If we consider that the agent may go bankrupt, this leads to the replacement of the agent, in which case a different investment scenario has to be used, perhaps one based on Entry/Exit multiplicative models (see Section 2.3.4).

(ii) If we consider an agent that owes money (has a negative budget value, $x(t) < 0$), then it can be seen that the dynamics in Eq. (2.9) lead to unreal situations in which negative returns result in gains and positive returns in losses. For example, consider an agent with $x(t) = -10$, $q = 0.1$ and $a(t) = 0$, i.e. the agent must borrow $I(t) = 1$ to perform an investment. If the return in the next step is positive with $r(t+1) = 1$, then from the dynamic of the investment model Eq. (2.9), we have $x(t+1) = -10(1.1) = -11$, which does not make sense. The same happens in the case of a negative return: for $r(t+1) = -1$, we have $x(t+1) = -10(0.9) = -9$, which is incorrect because for a negative return, the debt should actually increase to $x(t+1) = 11$. Thus, a different way to handle an agent owing money using the dynamics in Eq. (2.9), would be to consider an proportion of investment with the range $q \in [-1, 1]$. In this case, an agent with $q = -0.1$ would mean an agent with debts that decides to borrow an amount of money equivalent to 10% of its debts. Thus, assuming again an agent with $x(t) = -10$ and $q = -0.1$, for positive return $r(t+1) = 1$, we have $x(t+1) = -10(0.9) = -9$, i.e. the debt decreases as the agent wins money using borrowed money. On the other hand, for negative returns $r(t+1) = -1$, we have $x(t+1) = -10(1.1) = -11$, i.e. the agent loses the borrowed money and its debt increases.

(iii) We consider only the truncated versions of the distributions. For example the Truncated Gaussian distribution (see [Johnson et al., 1994, p. 156] and [Robert, 1995]):

$$p_{TN}(r) = \frac{\phi\left(\frac{r-\mu}{\sigma}\right)}{\sigma\left(\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right)} \tag{4.5}$$

where μ and σ are the mean and standard deviation of the returns, respectively, a and b are the lower and upper truncation points, respectively and $\phi(\cdot)$ and $\Phi(\cdot)$ are the probability density and cumulative distribution functions for the standard normal distribution, respectively.

Fig. 4.3 shows the most probable budget x_{mp} scaled by a/q^2, Eq. (4.4), for truncated Gaussian distributed returns $r(t) = TN(\mu, \sigma)$, Eq. (4.5), with $\mu = 0$ and different σ.

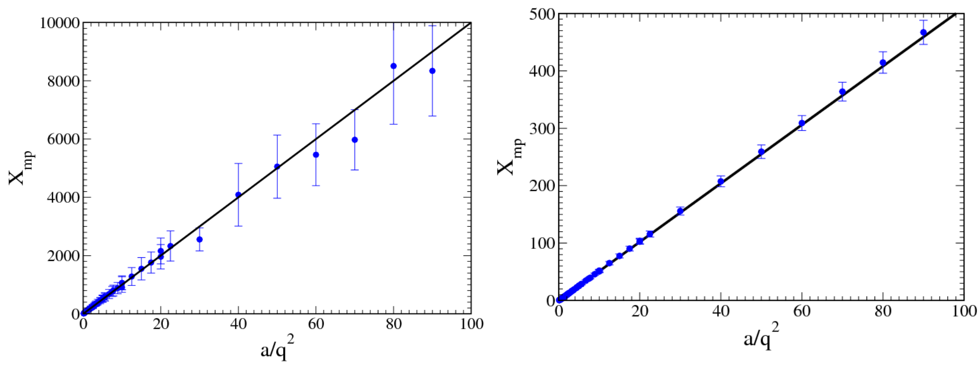


Figure 4.3.: Most probable budget value vs. scaled variable a/q^2 for truncated Gaussian distributed returns: (left) $r(t) \in TN(0, 0.1)$, the value found for the slope of the numerical simulations in this process was $c = 102.17 \pm 0.701$, and the analytical result $1/\langle r^2 \rangle = c = 100$ Eq. (4.8); (right) $r(t) \in TN(0, 0.5)$, with $c = 4.86 \pm 0.03$ and the analyical result $c = 5.1$. Further parameters as in Fig. 4.2.

For truncated ARCH or GARCH distributions, the situations are more complicated because to our knowledge, no closed formulas are available for the long-run variance of truncated ARCH/GARCH processes (here we refer to the literature [Bera and Higgings, 1993; Bollerslev, 1986; Engle, 1982; Nelson and Cao, 1992]). However, for comparison, in Fig. 4.4, we show the scaling obtained for ARCH and GARCH process with parameters resulting in small return values, where the outliers can be neglected (see Fig. 3.9).

4.4. Analytical Solution

In order to calculate the most probable value of the process, x_{mp}, which corresponds to the peaks of the distribution in Eq. (3.26), we proceed by deriving Eq. (3.26) and equalizing it

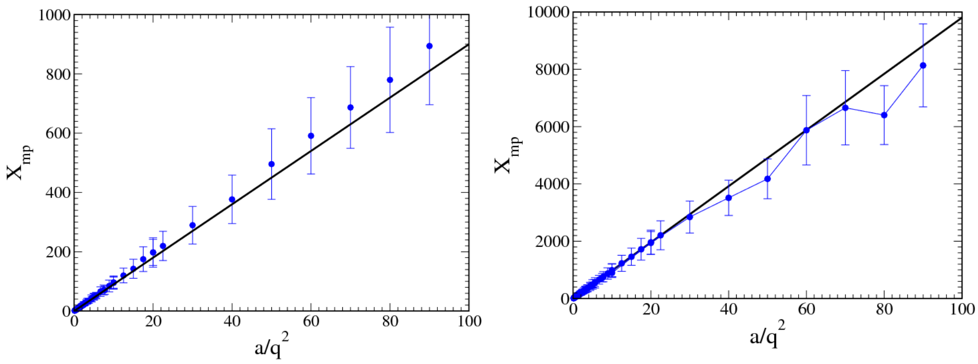


Figure 4.4.: Most probable budget value vs. scaled variable a/q^2 for (left) an ARCH(1) process ($\alpha_0 = \alpha_1 = 0.1$), with the slope $c = 9.089 \pm 0.029$ and (right) a GARCH(1,1) process ($\alpha_0 = \alpha_1 = \beta_1 = 0.01$), with the slope $c = 95.17 \pm 0.588$. Further parameters as in Fig. 4.2.

to zero as follows [Navarro-Barrientos et al., 2008b]:

$$\begin{aligned} 0 &\stackrel{!}{=} \partial_x P_s(x) \\ &= -\frac{1+\mu}{x}\, P_s(x) + \frac{a}{D_{\log\lambda}\, x^2}\, P_s(x) \\ &= \frac{P_s(x)}{x^2\, D_{\log\lambda}} \left[-D_{\log\lambda}(1+\mu)x + a\right]. \end{aligned}$$

Using the definition of μ in Eq. (3.25), we determine that the most probable budget can be described by:

$$x_{mp} = \frac{a}{D_{\log\lambda} - \langle \log\lambda \rangle}. \tag{4.6}$$

In our case, for $\langle\lambda\rangle = 1$, we have $\langle\log\lambda\rangle \approx 0$, and to the first order, this yields:

$$\begin{aligned} D_{\log\lambda} &= \left\langle \log^2\lambda \right\rangle - \langle\log\lambda\rangle^2 \\ &= \left\langle \log^2(1+q\,r) \right\rangle - \langle\log(1+q\,r)\rangle^2 \\ &\approx q^2 \left(\left\langle r^2 \right\rangle - \langle r\rangle^2 \right) \\ &= q^2\, D_r \\ &\stackrel{\langle r\rangle = 0}{=} q^2 \left\langle r^2 \right\rangle. \end{aligned} \tag{4.7}$$

Thus, a first approximation yields the following equation for the calculation of the most probable budget value x_{mp} given proportion of investment q, income a and returns $\langle r^2\rangle$ values:

$$x_{mp} \approx \frac{a}{q^2\, \langle r^2\rangle}. \tag{4.8}$$

The term $\langle r^2\rangle$ in this equation corresponds to the term c in Eq. (4.4). This is shown in the following where we analyse the resulting theoretical x_{mp} for different kind of returns.

If we assume that returns $r(t)$ are randomly drawn from the binomial distribution $B\{-1;1\}$, this yields:

$$\left\langle r^2 \right\rangle = 1 \tag{4.9}$$

$$\Rightarrow \quad x_{mp} = \frac{a}{q^2}. \tag{4.10}$$

If the returns are randomly drawn from the uniform distribution $U(-1,1)$:

$$\left\langle r^2 \right\rangle = \frac{1}{2}\int_{-1}^{1} r^2\, dr \tag{4.11}$$

$$= \frac{1}{3} \tag{4.12}$$

$$\Rightarrow \quad x_{mp} = 3\frac{a}{q^2}. \tag{4.13}$$

Note that these theoretical results agree with the empirical results shown in Fig. 4.2 for the scaling coefficient in Eq. (4.4), with $c = 1$ for binomial and $c = 3$ for uniform returns, respectively.

Now, if the returns are randomly drawn from the Gaussian distribution $N(0, 0.1)$:

$$\left\langle r^2 \right\rangle = \sigma^2 \;=\; 0.01. \tag{4.14}$$

However, as mentioned in Section 4.3.2, we need to consider truncated distributions in order to avoid bankrupcy or over-enrichment.

It can be shown that for returns drawn from a truncated normal distribution, i.e. returns in the range $r(t) \in (-1, +1)$, the second moment of the returns is given by:

$$\left\langle r^2 \right\rangle = \sigma^2 - \sigma\sqrt{\frac{2}{\pi}}\,\frac{\exp\left\{-\frac{1}{2\sigma^2}\right\}}{\operatorname{erf}\left\{\frac{1}{\sqrt{2}\sigma}\right\}}. \tag{4.15}$$

To elucidate the difference between discarding and truncating returns drawn from broader distributions in Fig. 4.5 (left), we show the value of $\langle r^2 \rangle$ for r drawn from a Gaussian distribution discarding returns outside the range $[-1, 1]$. Note that for $\sigma > 0.7$, r^2 starts to decrease in value, a fact which does not corresponds to the expected increase of the variance if the standard deviation also increases. On the other hand, Fig. 4.5 (right) shows the value of $\langle r^2 \rangle$ for r drawn from a Truncated Gaussian for the range $[-1, 1]$, Eq. (4.5). Note that when using the Truncated Gaussian for the calculation of $\langle r^2 \rangle$, the variance increases with respect to σ, as expected. Moreover, note that the theoretical coefficient for c in Eq. (4.4) for returns drawn from a Truncated Gaussian distribution $TN(0, 0.5)$ is $c = 0.5$, which agrees with the simulation results shown in Fig. 4.3.

If the returns are drawn from an ARCH(1)-process, the long-run variance of the ARCH(1)-process is known to follow the formula, see [Mantegna and Stanley, 2000, p.78]:

$$\sigma^2 \;=\; \frac{\alpha_0}{1 - \alpha_1}. \tag{4.16}$$

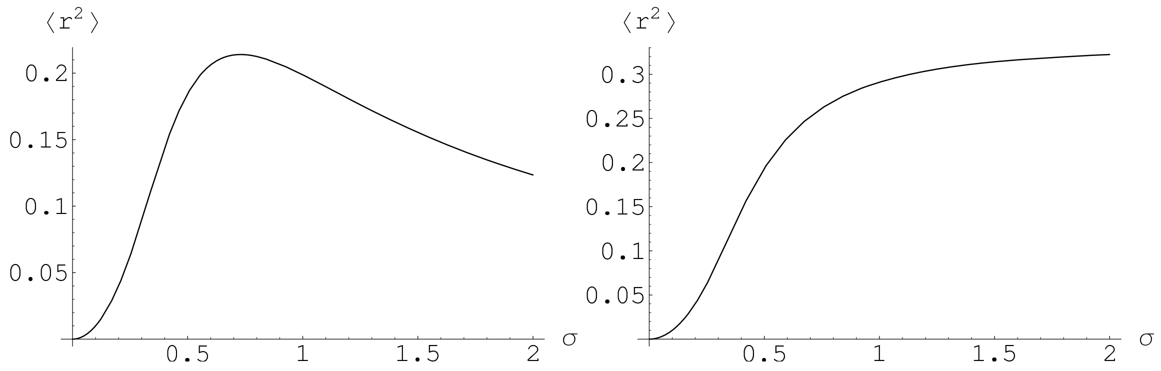


Figure 4.5.: $\langle r^2 \rangle$ for different σ values, where r is drawn from: (left) Gaussian distribution constrained to the range $[-1, 1]$ and (right) Truncated Gaussian distribution Eq. (4.5).

We find that with $\alpha_0 = \alpha_1 = 0.1$:

$$\left\langle r^2 \right\rangle = \sigma^2 \;=\; 0.111 \tag{4.17}$$

$$\Rightarrow \quad x_{mp} = 9\frac{a}{q^2}, \tag{4.18}$$

which agrees with the simulation results obtained in Fig. 4.4 (left).

Finally, for returns drawn from a GARCH(1,1) process, notice that the GARCH process is also defined on $\mathbb{R}$ and has zero mean, $\langle r \rangle = 0$. The long-run variance of a GARCH(1,1)-process is known to follow the following formula [Nelson and Cao, 1992]:

$$\sigma^2 \;=\; \frac{\alpha_0}{1 - \alpha_1 - \beta_1}. \tag{4.19}$$

For the case with $\alpha_0 = \alpha_1 = \beta_1 = 0.01$, we have $\sigma^2 = 0.010204$, which means that:

$$\Rightarrow \quad x_{mp} = 98\frac{a}{q^2}, \tag{4.20}$$

which agrees with the simulation results obtained in Fig. 4.4 (right).

4.5. Summary and extensions

From this chapter we learned the following:

It was shown that the number of time steps needed for the process to converge may vary depending on the type of returns.

We also review our approach to ensure stationary distributions and stationary most probable budget x_{mp} values based on hypothesis testing methods. A scaling function for the most probable budget x_{mp} was found (Eq. (4.8)). Using this function, an agent is able to calculate the x_{mp} given the external sources or income a, the desired proportion of investment q and the type of stochastic returns r. Interestingly, if returns are drawn from distributions with zero mean and that are covariance-stationary, the most probable budget value x_{mp} can be determined by means of calculating the variance of the returns for any fixed income and fixed proportion of investment values. Finally, we compared the theoretical scaling function with the scaling function fitted to computer experiments, leading to an agreement for returns with a pull to zero.

Note:

The agent can use the scaling function in Eq. (4.8) not only to determine the most probable budget value given the properties of the retuns, the proportion of investment and the incomes, but also to know the consequences in terms of most probable budget for any changes in these parameters. For example, given a_1 and q_1 parameter values for income and risk, respectively, if one of these values changes and if the agent wants to achieve the same most probable budget value, then the proper change in the parameter values is calculated as follows. Assuming the new parameter values are a_2 and q_2, if, for example, the income of an agent has changed from a_1 at time step t to a_2, the agent may like to change its proportion of investment to

$$q_2 = \sqrt{q_1^2 \frac{a_2}{a_1}}, \tag{4.21}$$

in order to mantain the same x_{mp}. In the same manner, if the agent decides to change the proportion of investment, then in order to gather the same x_{mp}, the new income a_2 needs to be:

$$a_2 = q_2^2 \frac{a_1}{q_1^2}. \tag{4.22}$$

Furthermore, assume that the external income of the agent has changed at time step t by a ratio R_a so that the new income is $a_2 = R_a\, a_1$. The agent may be willing to increase/decrease the proportion of investment accordingly to this increment/decrement on the external incomes to $q_2 = \sqrt{R_a}\, q_1$ in order to receive the same x_{mp} that it was receiving before. Note that these relations are true for returns that lead to stationary budget distributions.

Further work:

It would be interesting to find a scaling function for other type or returns, like those that can be drawn from simple and geometric Brownian motion and different ARCH/GARCH processes like: exponential GARCH (EGARCH), in which the model is based on a logarithmic expression of the conditional variability [Nelson, 1991] and the Threshold ARCH (TARCH,) which is an asymetric model based on the assumption that unexpected changes in the returns have different effects on the conditional variance [Zakoian, 1994]. Finally, following [da Silva et al., 2005], it might be also interesting to analyse the persistance in the model by measuring the chance of keeping a positive balance relative to the initial amount invested. Similarly to the scaling law function found for most probable budget value in our investment model, the budget persistance may be described by a scaling function for different q and a values.

5. Constant Proportions of Investment for Real Returns

This chapter analyses the dynamics of the budget for fixed investment strategies in returns from stock market data.

5.1. Introduction

So far, we have considered returns on investment either drawn randomly from different known probability distributions or generated using a stochastic process with correlations in time. The main goal of this chapter is to investigate the evolution of the budget of the agent for different fixed proportions of investment and the influence of the external sources for real returns.

In Chapter 3 and in Chapter 4, we analyzed the budget evolution and the budget scaling for a fixed constant proportion of investment and fixed incomes for random returns drawn from binomial, uniform and Gaussian distributions and also for returns modeled by random processes like the ARCH and GARCH processes. Now, in this chapter we investigate the influence of different fixed proportion of investment and fixed incomes on the budget distribution, its evolution over time and the scaling function for the budget for real returns from stock market data. This chapter is organized as follows: in Section 5.2, we present the data that we gather from the stock market to model the real returns on investment. In Section 5.3, by means of simulations, we analyze the dynamics of the budget over time for different constant proportions of investment and different income values. Afterwards, in Section 5.3.2, a scaling function for the most probable budget is discussed.

5.2. Properties of the Real Returns

For the real returns data from approximately 6686 stocks from the US stock market was collected for the daily closing prices with range dates varying between 08/04/1994 to 16/04/2004. Consider the following scenario: an agent buys a number of stocks s at day $t-\tau$ at the price $p(t-\tau)$, keeps the stocks for a given period of time $\tau \geq 1$ and sells them at day t at the price of $p(t)$. In the literature, several approaches to calculate the stock price change rate can be found (see [Mantegna and Stanley, 2000, p.37]). For simplicity, we assume the following calculation for the return on investment, which provides a direct percentage of gain or loss:

$$r(t) = \frac{p(t) - p(t-\tau)}{p(t-\tau)}. \tag{5.1}$$

This means that for $t-\tau$ time steps the agent holds a number of shares $s(t)$ of a given stock, i.e. the agent has bought the number of shares $s(t)$ at time step $t-\tau$ for the price $p(t-\tau)$ and may be willing to sell them at time t for the price $p(t)$. For simplicity, the agent receives returns from buying and selling an amount of stock only, i.e. it is not taken into account that the corporation gives a share of net income in the form of dividends to the agent.

Thus, in terms of the proportion of investment $q(t)$, this means that at time step t, the agent owns the following amount $s(t)$ of shares of a given stock [Anufriev and Bottazzi, 2006]:

$$s(t) = \frac{x(t)\,q(t)}{p(t)}. \tag{5.2}$$

For example, assume an agent with budget $x(0) = 100$ and an proportion of investment of $q = 0.1$. This means that the agent decides to invest 10% of its budget, i.e. invests the amount $I(0) = 10$. In terms of buy-sell actions, if $\tau = 1$, i.e. daily RoI, and the price of stock share is $p(0) = 1$, by investing $I(0) = 10$ the agent is actually buying 10 shares of stock at time $t = 0$, which means that the agent owns $s(0) = 10$ shares. Now, if the price of the stock for the next time step increases to $p(1) = 2$ and the agent decides to sell all its shares using Eq. (5.1), this would yield a return of $r(1) = 1$ for the agent, and its wealth would increase to $x(1) = 120$. However, if we assume that the agent has a constant proportion of investment, this means that actually the agent is not selling all the shares, but only a part of them. Thus, for this case, the agent needs to invest $I(1) = 12$, which for this approach, yields the following two possible actions for the agent: (i) to sell 4 shares only and to keep the rest $s(2) = 6$ shares and (ii) to sell all shares and invest $I(1) = 12$ by buying shares from other stocks. Note that both buy and sell orders that are placed in the market are always fulfilled and have a fixed price, i.e. the agent is able to buy and sell at time t as many shares as it wishes at the price of $p(t)$.

Before we start analyzing the dynamics of the investment model for real returns when using constant proportions of investment, we first need to define how to treat the approximately 12×10^6 returns that we have already obtained from the prices of the stocks. The following approaches could be used:

1. All the returns are saved in an array and for every time step t, a return is uniformly randomly drawn from this array. This is the equivalent of finding the distribution of the returns and drawing randomly at each time step a return from this distribution. Obviously, using this approach means that returns have no correlations in time and that there are also no correlations between stocks. Moreover, it may be interpreted as a method in which an agent randomly chooses when and where to invest its money and holds the stock for only one time step $\tau = 1$ in Eq. (5.1). In other words, at every time step, the agent sells shares from stock i and buys shares from stock j, where i and j are not necessary different.

2. A more realistic approach would be to consider returns with correlations in time. This can be achieved by assuming that the agent randomly chooses a stock, buys the stock at time step t' at the price of $p(t')$ and holds the stock for τ time steps. Later on, at time step $t = t' + \tau$ the agent sells the stock at price $p(t)$ receiving its corresponding RoI.

For simplicity, the former approach is considered for the simulations, i.e. daily returns with no correlations in time, which means that the agent holds the stock for only one time step.

Firstly, we need to illustrate the basic features of the RoI for our data from the US stock market. Fig. 5.1 (left) shows the evolution of the distribution of the returns over time in a semi-logarithmic scaled plot. For this, the daily returns were calculated and saved in an array. Afterwards, the returns were drawn randomly uniformly from the array with all possible returns, not accounting for time correlations or types of stock (the agent randomly chooses when and where to invest its money). Note in Fig. 5.1 (left) that if more returns are drawn from the array, the distribution of the returns becomes thinner, which may be described with a Lévy stable distribution (see [Mantegna and Stanley, 1994] and [Mantegna and Stanley, 2000] (chapter 8 and 9)). A visual impression indicates that the distribution of the returns in Fig. 5.1 (left) is symmetrical and centered at zero; however, for larger t, it seems that positive values are more probable than negative values. For this, Fig. 5.1 (right) shows the evolution of the average value of the returns over time. Note that, the interval of data for the real returns that were gathered leads on average to larger positive than negative return values with $\langle r \rangle \approx 0.0009$. This is in agreement with some investigations where a typically small positive mean value of returns is reported for some international stock markets [Bertram, 2004; Gorski et al., 2002; McMillan, 2005].

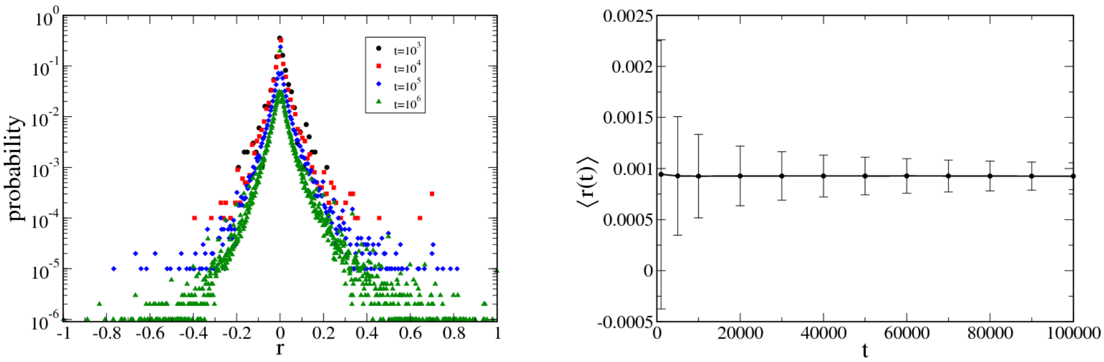


Figure 5.1.: Evolution over time of real returns, Eq. (5.1): (left) distribution and (right) average value. For returns drawn randomly from data from the US stock market, Eq. (5.1), with $\tau = 1$ and with range dates varying between 08/04/1994 to 16/04/2004. Further parameters as in Fig. 3.1

As we discussed in Section 2.3.3, one of the factors responsible for stable power law distributions is the presence of a pull to zero values. As you can see, if $\langle r \rangle > 0$, this means that there is no pull to zero values and that the distributions of the budget will go to $+\infty$ values over time, i.e. continual shifting of the distribution to larger positive values. This is shown in Fig. 5.2 where the distribution of the budget does not converge to a stationary state distribution.

Based on these results, we notice that we can enforce the pull to zero values if we include transaction costs in the calculation of the returns; this also presents a more realistic scenario. By transaction costs, we mean the costs of buying and selling shares of a stock. It is well known that these costs can vary depending on the market and the number of shares that are bought or sold. However, in most of the cases, the transaction costs charged when the agent bought or sold shares based on a fixed percentage, commonly from 0.5 to 1 percent of the market value.

For simplicity, the percentage transaction costs are described by a fixed ratio k from the price of the share in the market. In order to include the transaction costs k in the calculation of the returns in Eq. (5.1), for the sake of clarity, let us first consider them in

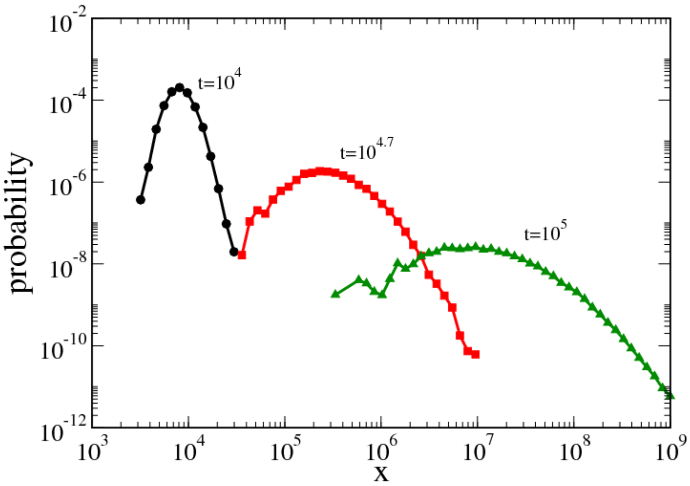


Figure 5.2.: Evolution of the budget probability distribution over time for proportion of investment $q = 0.1$ and income $a = 0.5$ for real returns as specified in Fig. 5.1. Further parameters as in Fig. 3.1

the investment model given by Eq. (2.9). For this, note that a transaction cost has to be considered for each buy and sell action performed over time. For $\tau = 1$ and in accordance with Eq. (5.2), the number of shares $s(t)$ that the agent owns at time t can be described by:

$$s(t) = \frac{q(t)\, x(t)}{p(t)(1+k)}. \tag{5.3}$$

It can be shown that by including transaction costs, described as a percentage k of the price of the stock, and using both the alternative description for the investment model described in Eq. (2.12) and the number of shares owned by the agent shown in Eq. (5.3), the investment dynamics can be now described by:

$$x(t+1) = x(t) - q(t)\, x(t) + s(t)\, p(t+1)\, (1-k) + a(t). \tag{5.4}$$

Note that the second term corresponds to the money paid for buying $s(t)$ shares at time step t and the second term corresponds to the amount that will be received for selling $s(t)$ shares at time $t+1$ for the price of $p(t+1)$ plus transactions costs. After doing some algebra, it can be shown that the investment dynamics can be described by:

$$x(t+1) = x(t) \left[1 + q(t) \left(\frac{p(t+1)\,(1-k)}{p(t)\,(1+k)} - 1\right)\right] + a(t). \tag{5.5}$$

Rewriting Eq. (5.5), it can be shown that the dynamics can be expressed using Eq. (2.9) with returns described now by:

$$r(t) = \frac{p(t+1)\,(1-k) - p(t)\,(1+k)}{p(t)\,(1+k)}. \tag{5.6}$$

For the sake of clarity, consider the following example: assume an agent with budget $x(0) = 100$, constant proportion of investment $q = 0.1$ and fixed income $a = 1$. This means that the agent is willing to invest $I(0) = 10$ initially. If the price of a share of stock i is $p_i(0) = 1$ and the transactions costs represent 1% of the price, i.e. $k = 0.1$, the agent buys $s(0) = 9.09$ shares. For simplicity, we avoid rounding the amount $s(t)$ to an integer number of shares, in other words, we assume that the agent can also buy a fraction of a share (maybe an agent buys a share together with other agents). Thus, at time step $t = 1$, if the price of the share of stock i is now $p_i(1) = 2$, and if the agent sells all its shares, then the budget of the agent at time step $t = 1$ would be $x(1) = 106.36$. As it was previously noted, we assume for simplicity that the agent sells shares from stock i and buys shares from stock j at every time step. This means that in our example, at time step $t = 1$, the agent with budget $x(1) = 106.36$ is willing to invest $I(1) = 10.63$ and repeats the whole process by again choosing a stock, in this case stock j, and buying $s(1)$ shares to the price $p_j(1)$.

Now, as discussed above we need to determine which percentage transaction costs are more suitable to fulfill the condition of a pull to zero needed to obtain power law distributions of the budget. For this, Fig. 5.3 (left) shows the average return for different percentage transaction costs and Fig. 5.3 (right) shows the average return value over time for the percentage transaction costs $k = 0.00046$. It can be seen that this latter leads to the desired property $\langle r \rangle \approx 0$, which fulfills the pull to zero (see Section 2.3).

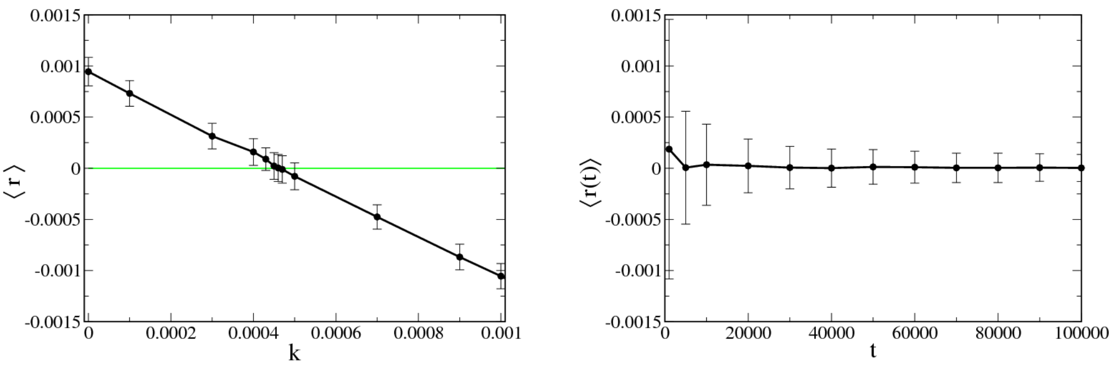


Figure 5.3.: Average return value: (left) for different percentage transaction costs k, returns obtained after $t = 10^5$ time steps; and (right) for the percentage transaction costs $k = 0.00046$, evolution over time. Simulations for $N = 100$ trials.

5.3. Simulation Experiments

Here, we present stochastic computer simulations of Eq. (2.9) for different distributions of $r(t)$. Initially, $x(0) = 10$ holds for the agent's budget, $q(t) = q$ and $a(t) = a$ are kept constant during each simulation. The distributions were realized for $N_a = 10^4$ trials.

As in previous chapters, in order to elucidate the dynamics of the budget, the evolution of the probability distribution of the budget $p(x, t)$ over time and the average of the most probable budget value $\langle x_{mp}(t) \rangle$ are investigated by means of computer simulations.

5.3.1. Budget Evolution

A graphical visualization of the evolution of the budget distribution assuming transaction costs can be seen in Fig. 5.4. For the simulations, we assumed a fixed income of $a = 0.5$ and a fixed proportion of investment of $q = 0.1$. Note that the distribution reaches a stationary state after approximately $t = 10^5$ time steps.

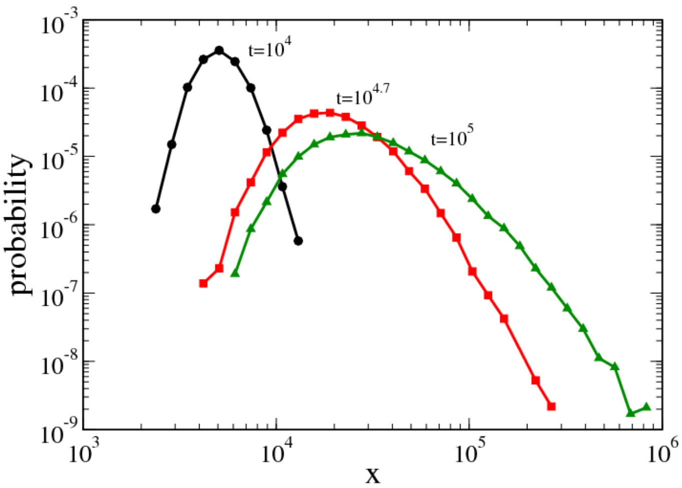


Figure 5.4.: Evolution of the budget probability distribution over time for proportion of investment $q = 0.1$ and income $a = 0.5$. For returns from the US stock market Eq. (5.6) with percentage transaction costs $k = 0.00046$. Further parameters as in Fig. 3.1.

Again, we note that in order to obtain the average most probable budget value $\langle x_{mp} \rangle$ for different income and proportion of investment values, we need to determine the number of time steps t_s needed to find stationary distributions. For this, we used Algorithm 2 (Section 3.2.3) and we show in Fig. 5.5 the evolution of the x_{mp} over time for RoI with transaction costs $k = 0.00046$, Eq. (5.6). Fig. 5.5 shows the evolution of x_{mp} for (left) constant proportion of investment and different income values, and (right) constant income and different proportion of investment values.

For all different environments treated in this section, we show that for small proportion of investment values, the distribution of the budget needs larger number of time steps to reach a stationary state. The reason for this is probably that the *pull to zero* condition needed for power law distributions in the dynamic of multiplicative stochastic process with an additive term is not strong enough.

5.3.2. Budget Scaling for Proportion Of Investment and Income

In Chapter 4, it was shown that the most probable budget value x_{mp} can be described by the scaling function in Eq. (4.4), which describes x_{mp} for the proportion of investment, incomes and the second moment of the returns. In Chapter 4, computer experiments and analytical solutions for the scaling function were presented and discussed for returns drawn from known distributions and stochastic processes with correlation in time. In this section,

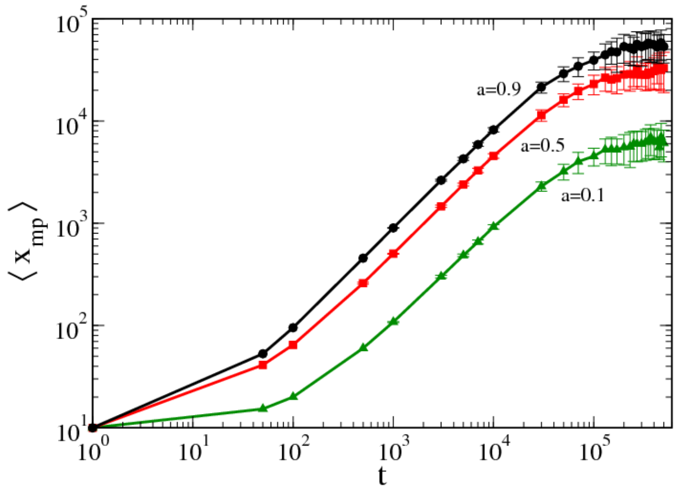


Figure 5.5.: Most probable budget value for RoI from the stock market, Eq. (5.6), with percentage transaction costs $k = 0.00046$. For constant proportion of investment $q(t) = 0.1$ and different income values a. Further parameters as in Fig. 3.3.

we investigate this scaling function for real returns. For this we use data from the stock market, introduced in Section 5.2. Consider Eq. (5.6) for the calculation of the returns and percentage transaction costs $k = 0.0046$. Fig. 5.6 shows the scaling function of x_{mp} for different q and a values.

In Section 4.2, it was mentioned the fact that the number of time steps needed to reach stationary distributions may depend on the properties of the returns. Algorithm 2 (Section 3.2.3) presented a way to deal with this problem. This problem may be more acute for real returns specially for small q and large a because of the weak pull to zero values as seen in Fig. 5.6, where some simulation results do not fit a straight line very well. For clarity, we investigated the number of time steps t that were needed to reach stationary budget distributions for real returns, Eq. (5.6), with transaction costs $k = 0.0046$ for different q and a values. The resulting number of time steps is shown in Fig. 5.7. As we noted previously, it is clear that the number of time steps needed to reach an stationary state is much larger for small values of q; also observe that a does not have a clear influence on the number of time steps needed for convergence.

The previous results show that the number of time steps t_s needed to reach a stationary distribution depends much more on the value of q and the distribution of the returns r than on the income a. A more exact determination of the number of time steps needed for the process to converge to stationary distributions may be obtained following the analytical approximation proposed by [Sornette and Cont, 1997] for large but finite t: $x(t) \approx \sqrt{D\,t}$ for D as in Eq. (2.20). However, as previously mentioned, for practical purposes, in our investigations we use Algorithm 2 (Section 3.2.3) to determine the point at which the process has converged to a stationary distribution.

Finally, Fig. 5.8 shows the evolution of the $\langle r^2 \rangle$ of the returns over time. Note that this value leads to an experimental value of $\langle r^2 \rangle \approx 0.001675$, which means a coefficient of $1/\langle r^2 \rangle = c = 597.01$ for the scaling function in Eq. (4.4), which agrees with the scaling

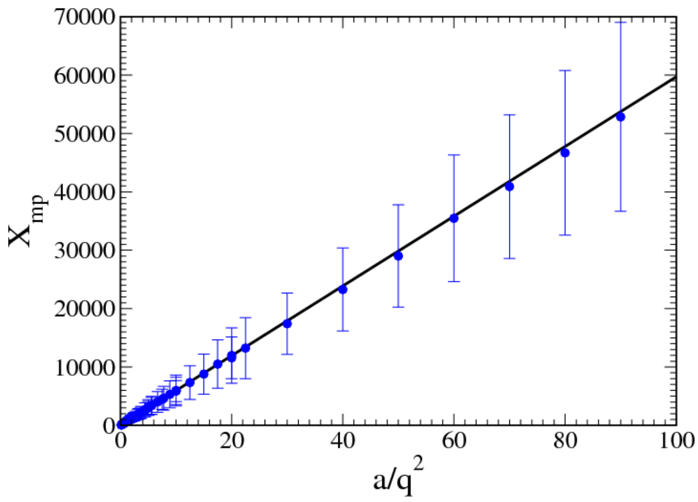


Figure 5.6.: Most probable budget value vs. scaled variable a/q^2 for real returns with transactions costs $k = 0.00046$ Eq. (5.6). The value found for the slope of the numerical simulations was $c = 599.692 \pm 1.001$.

found in the simulations of the dynamics shown in Fig. 5.6.

5.4. Summary and Extensions

From this chapter we learned the following:

For real returns from the stock market with transaction costs, if the agent decides to invest a fixed percentage of its budget, we find that the distribution of its budget converges over time to a stationary distribution with a power law in the tail. However, this is true only for a fixed range of transaction costs values that are below the standard transaction costs in real life. It was also shown that the number of time steps needed for the process to converge to stationary distributions variates depending on the investment percentage of the agent. This result implies that for sufficiently low transaction costs, the budget distribution of the agent may reach a stationary state or it may shift to larger positive values over the course of time. Although these results do not mirror reality because normally the transaction costs are much higher, these results provide some insight into the dynamics of the investment model for real returns and help to realize the importance of the properties of pull and repulsion from zero in the dynamics of multiplicative stochastic processes, in this case for real data from the stock market.

Further work:

It would be interesting to find a theoretical scaling function that agrees with returns that do not pull the process to zero values, i.e. for costs different from those considered in this chapter or no percentage transaction costs. In this chapter it was considered that the agent randomly chooses when and where to invest its money, holding the stock for only one time step. As noted in Section 5.2, this approach means that returns have no correlations in time and that there are also no correlations between stocks. It would be interesting to extend this approach constraining the agent to invest according to the time line of the stock prices.

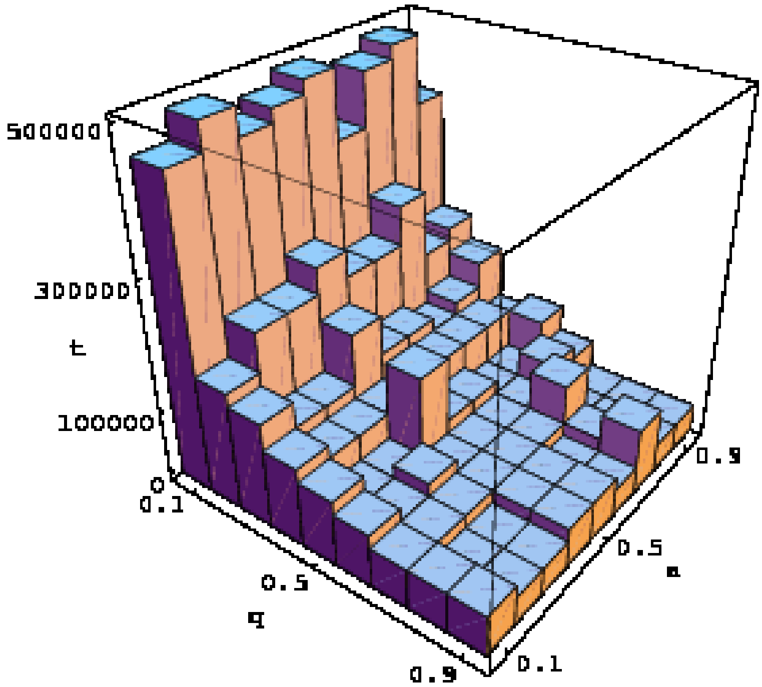


Figure 5.7.: Number of time steps t needed to reach stationary budget distributions for real returns, Eq. (5.6), with transaction costs $k = 0.0046$ for different proportions of investment q, different incomes a. Further parameters: $pv = 0.1$ in Algorithm 2 (Section 3.2.3).

Finally, it would be also interesting to consider not only daily returns, but short and long term investment scenarios as well.

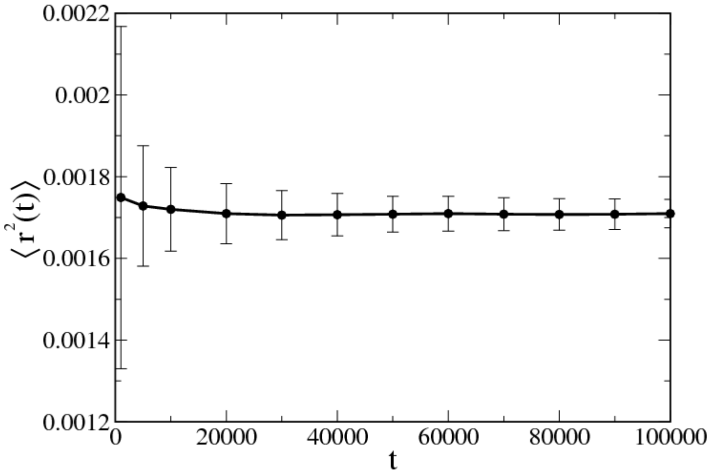


Figure 5.8.: For real returns with transactions costs $k = 0.00046$, Eq. (5.6), the average of the squared returns over time converges to the value $\langle r^2 \rangle \approx 0.0017$. Average over $N = 10^4$ trials.

Part II.

Adaptive Investment Strategies and Formation of Common Investment Networks

6. Investment Strategies for Stationary Noisy Periodic Environments

This chapter presents different agent strategies for an artificial investment scenario where the return on investment is characterized by a periodic function with different types and levels of noise. Two behaviors for the agent are considered: cautious behavior in which the agent chooses an investment proportional to the expected return and a daring behavior in which the agent chooses to perform large investments if the expected return is positive. In addition to these different strategies, the agent may have different capabilities to predict the future returns depending on its internal complexity.

6.1. Introduction

In the previous chapters we analyzed the dynamics of the investment model Eq. (2.9) for different constant proportions of investment and constant incomes. In our approach an agent was assigned a fixed investment proportion value for all time steps, i.e. the agent was not able to change or update its proportion of investment. Now in this chapter we assume that the agent is able to change the proportion of investment over time. The manner in which the agent changes this may depend on different assumptions that the agent may make about the dynamics of the environment.

Recall that one of the main goals of this PhD thesis is to investigate the extent to which internal complexity of agents influences their overall performance. For this, Fig. 1.2 depicted in a schematic diagram the three different types of agents that are of interest in this PhD thesis: reactive agent, experience-based agent and machine-learning-based agent. The dynamics for a reactive agent in random environments was studied in Chapters 3 and 4. Now, in this chapter, some strategies for experienced and machine-learning based agents are presented and in the following chapter, their performance is analyzed for periodic returns. But first, some approaches found in the literature that deal with the problem of finding a proper investment strategy in an uncertain environment are presented.

To find a proper method to control the proportion of investment is a complex and difficult task since most investment environments are uncertain and present fluctuations. For example, choosing to avoid investing may lead to losing big opportunities to win large amounts of money. On the other hand, choosing to invest large amounts of money may lead to situations where the chances of losing the complete budget are high. Thus, the task of finding a good strategy that controls the proportions of investment balancing between these two extrema is by far not trivial. In economics for example, this problem usually concerns the behavior that an investor should follow in order to maximize profits within an uncertain environment. To this end, researchers usually investigate the relationship between methods for optimization under uncertainty, the different preferences of an investor and the amount of information available from the environment. It is also important to mention that the golden rule of any economic endeavor is to try to maximize the profits or at least to minimize the losses as much as possible [LeCorre and Mischke, 2005]. Maybe one of the most important papers regarding the problem of maximization of profits in an uncertain environment is the seminal paper of Kelly [1956] where the author assumes a gambler performing some bets, in terms of a fraction of his capital, on a given random process. Kelly shows that by knowing the probability of winning and speculating the result of the bets in advance, the gambler can maximize the exponential rate of his wealth, Eq. (2.6). However, this approach makes sense for bets with a higher rate of winning than losing, because if we consider a fair bet with equal probability to win or to lose, then according to Kelly it is better not to bet at all.

On the other hand, many researchers have considered machine learning methods to find good investment strategies in different type of stochastic environments, some contributions have been already mentioned in Section 1.2.1. From these contributions, it is important to note some of their disadvantages. For example, *neural networks* have been one of the preferred methods of many researchers. However, [Zell, 1995, p. 137] noted that the use of simple feed-forward neural networks for the prediction of time series have the disadvantage that only fixed window input sizes can be considered. In practice, the usual approach is to train neural networks with different topologies, however, fast training algorithms are needed to cover a large number of different topologies [Castiglione, 2004]. In the area of *reinforcement learning*, to our knowledge a direct application of algorithms like Sarsa and Qlearning (see [Sutton and Barto, 1998]) for investment strategies in uncertain scenarios has not been proposed yet. In this chapter we investigate in Section 6.6.1 a simple strategy based on reinforcement learning which is called “Iterative Update Rule” for the sake of simplicity. In the area of *evolutionary computation* the method that is often used by many researchers is *genetic programming*, however, its performance depends on the mathematical operations and mathematical functions considered and once an optimal strategy is found, it may be difficult to understand why does it work (see [Schulenburg and Ross, 2001]). The performance of reinforcement learning and genetic programming approaches for trading in the foreign exchange market has been investigated by [Dempster et al., 2001], in which the authors show that genetic programming outperforms the other strategies. From these different approaches used for prediction and finding good investment strategies, we note that there are not many approaches for investment strategies based on the standard generational *genetic algorithm*. Furthermore, note that some researches have investigated the use of standard generational genetic algorithms in changing environments [Branke, 1999; Grefenstette, 1992]. Because of this, in this PhD thesis, an approach based on standard generational *genetic algorithms* for the problem of investment optimization is proposed and its performance is compared with the performance of other investment strategies for different scenarios. For this, a typical scenario to study investment strategies is to let an agent choose between investing in a risk-free asset or in a risky asset (see [Arrow, 1965; Pratt, 1964; Tobin, 1958]). This approach is usually extended to more complex models where agents interact with each other by means of selling and buying goods. Examples for this are those contributions where an artificial stock market is simulated in which some of the stylized facts present in real markets are observed, see [Farmer, 2001; LeBaron, 2000; Lux and Marchesi, 2002; Raberto et al., 2003]. Some other researchers have compared the performance of different strategies in these type of scenarios. For example the investigations done by [Farmer et al., 2005; Gode and Sunder, 1993] where zero-intelligence agents and rational agents are compared. However, before investigating the performance of investment strategies in these more complex scenarios, it is necessary to understand their dynamics and to investigate their performance in a simpler scenario like the one presented in Section 2.2.

Thus, in this chapter and the following chapter, the following research question is addressed: to what extent is it valuable or recommendable to have an agent with complex learning mechanisms instead of a reactive agent for stylized exogenous returns? To answer this question, in this chapter, the different investment strategies and different agent's attitudes are presented. Afterwards, in Chapter 7 the performance of these strategies and attitudes is investigated in an simple investment scenario with noisy periodic returns.

This chapter is organized as follows. Firstly, Section 6.2, presents the investment scenario, i.e. the properties and abilities of an agent acting/investing in this environment. Secondly, in Section 6.4 the strategies used as a reference for the performance comparison are described. Finally, Section 6.5 and Section 6.6 show technical analysis and machine learning approaches for finding proper investment strategies.

6.2. Investment Scenario

As a necessary step to study more complex scenarios, we are interested in an initial study of the performance of different agent architectures/investment strategies in the investment scenario presented in Section 2.2. In this simple scenario, we consider a representative agent model for learning from the interaction with different type of time series. As explained in Section 2.2, this means that, in this approach, the agent invests independently in the market, i.e. there is no interaction or communication with other agents. Also, agents do not generate feedback with their investments on the market. In other words, the *environment* of the agents is not influenced by their investments.

In Chapters 3 and 4, fixed external incomes $a(t) = a$ were considered for the agent, Eq. (2.9). It was shown in Sections 2.3.1 and 2.3.2 that in the presence of random returns, an absence of these external incomes lead the agent to a bankrupt state. However, if the agent now has some methods to forecast the next return and the returns are somehow predictable, then the agent does not need external incomes to avoid bankruptcy. Thus, for the sake of simplicity, we assume no external incomes for the investment model:

$$x(t+1) = x(t)\left[1 + r(t)\,q(t)\right], \tag{6.1}$$

where, for completeness, $x(t)$ represents the budget of the agent, the market is expressed by $r(t)$, the return on investment, *RoI* and the agent has to choose between investing in a risk-free asset or in a risky asset by means of the proportion of investment variable $q(t)$. In other words, based on the previous received returns the agent may have a preference to keep cash (liquidity preference) or to invest in the market (speculative preference). Moreover, the behavior of the agent towards risk is assumed to be a *risk-neutral behavior*, i.e. the agent estimates only the expected return and based on this estimation, decides to increase or decrease its proportion of investment over time. Note that, in this chapter, we do not construct and investigate a market model; rather, our focus lies on investigating the characteristics of good and bad strategies, from the point of view that the agent gathers previous returns to perform some forecasting. Thus, the challenge for the agents is twofold: first, agents have to predict $r(t)$ as accurately as possible, and second, they have to adjust $q(t)$ to the proper values as quickly as possible. Thus, to determine the most appropriate investment by an agent at a particular time, methods from technical analysis are considered, as well as a selection of other methods of varying complexity from various fields.

As explained before, we are interested in how the market dynamics, $r(t)$, affect the different investment strategies of the agent, $q(t)$. It is very important to realize that the market dynamics – while affecting each agent's $q(t)$ – are *not completely known* to the agents. This means that at time t each agent only receives the *actual value* of the Return on Investment (RoI) and adjusts its proportion of investment accordingly, *without* having a complete knowledge about the dynamics of $r(t)$. The agent may, of course, have some bounded memory about past RoI that could be used for predictions of future RoI. However, the agent has to gather information about the ups and downs of the RoI and to draw its own conclusions from this information by itself. Therefore, the investment strategy that better guesses about the dynamics of $r(t)$ will of course perform better in the environment.

6.3. The Periodic Return on Investment

In these investigations, periodic *returns on investment (RoI)* are assumed, the dynamics of which are controlled mainly by a periodicity and some noise as follows [Navarro-Barrientos et al., 2008a]:

$$r(t) = A \sin(w\,t + \xi_1) + \xi_2, \tag{6.2}$$

where the amplitude A of the sinusoidal function depends on the amplitude noise level $\sigma_2 \in (0,1)$, with $A = 1 - \sigma_2$; the frequency w depends on the periodicity T (in Section 8.4.1 a non-fixed periodicity is considered); the phase ξ_1 is a random number drawn from a Uniform distribution $\xi_1 \in \pi U(-\sigma_1, \sigma_1)$ with $\sigma_1 \in (0,1)$ and ξ_2 corresponds to a random number drawn from a Uniform distribution $\xi_2 \in U(-\sigma_2, \sigma_2)$. Thus, σ_1 and σ_2 account for the fluctuations in the market dynamics, on the phase and the amplitude of the RoI, respectively. Note that the noise parameter $\sigma_{1,2}$ gives us a way of controlling the noise in the RoI, thereby allowing us to evaluate the various strategies for different scenarios, ranging from a completely clear signal with no noise at all (for $\sigma_{1,2} = 0$) to a noise-only signal (for $\sigma_{1,2} = 1$). This makes it possible for us to determine how well multiple strategies perform for different types and levels of noise and what impact the type and level of noise has on a single strategy.

Fig. 6.1 shows an example of these two kind of RoI for different noise levels σ_1 and σ_2.

In order to interpret some of the results obtained in the next chapter, it is useful to understand some properties of the periodic returns. An important feature of the returns is the probability distribution, which is shown in Fig. 6.2. It can be observed that the distribution of the returns for fluctuations on the phase does not change as the level of noise is increased, whereas for fluctuations on the amplitude there is a notable change as the level of noise is increased. More precisely, for the probability distributions of $r(t)$ with phase noise, we see that there is a higher probability of values close to -1 and 1, and a lower probability of values close to 0. Note that this is the same distribution that is found for a sine wave with no noise at all. Thus, for phase noise, the value of σ_1 has no effect on the distribution of the returns. For the probability distributions of $r(t)$ with amplitude noise, we observe a probability distribution which is a combination of the probability distribution of a sine wave without noise (caused by the sine wave) and a uniform probability distribution (caused by the noise). For higher levels of noise, the convolution of probability distributions more closely resembles the uniform distribution, and for lower levels of noise, the convolution of probability distributions more closely resembles the sine wave distribution. Thus, for amplitude noise, the value of σ_2 is crucial and different values lead to different distributions.

Two properties that are also of particular interest are the absolute average value of the

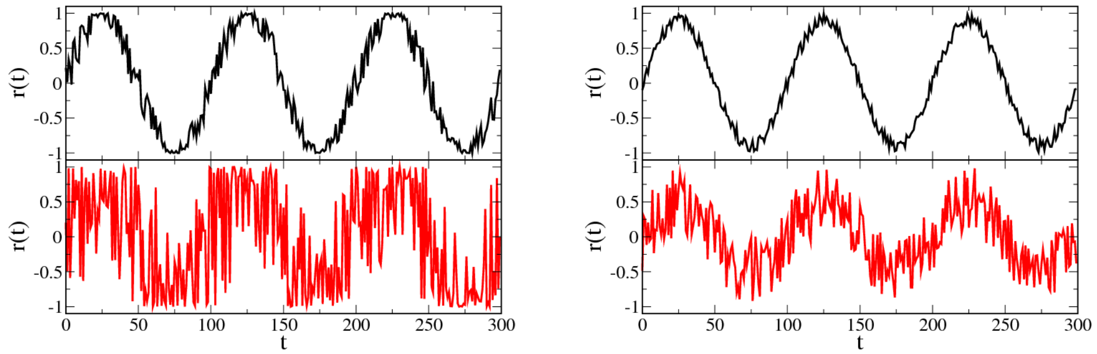


Figure 6.1.: Periodic RoI $r(t)$, Eq. 6.2, for: (left) no amplitude fluctuations, $\sigma_2 = 0$, and two different noise on the phase levels: (top) $\sigma_1 = 0.1$ and (bottom) $\sigma_1 = 0.5$; (right) no phase fluctuations, $\sigma_1 = 0$, and two different amplitude noise levels: (top) $\sigma_2 = 0.1$ and (bottom) $\sigma_2 = 0.5$. Further parameters: $T = 100$.

return and the correlation between the sign of two consecutive returns.

The average absolute RoI is of importance because its known from multiplicative stochastic processes [Navarro-Barrientos et al., 2008b; Sornette and Cont, 1997], that for a constant investment $q(t) = q_0$, the better-performing constant strategies are the ones that invest the least possible amount. Since $q(t)$ is multiplied with $r(t)$ in Eq. (6.1), the change in average absolute value of $r(t)$ has an impact similar to the change in $q(t)$ seen in the multiplicative stochastic processes, see Fig. 3.6. This leads to changes in performance that are not necessarily related with the performance of agents, and should be taken into account when interpreting the results.

Analytically, the average absolute value of the returns, $\langle |r(t)| \rangle$, can be calculated as follows:

$$\langle |r(t)| \rangle = \frac{1}{T} \int_0^T r(t)dt = \frac{2}{T} \int_0^{T/2} r(t)dt. \tag{6.3}$$

For returns with no noise $\sigma_1 = 0$ and $\sigma_2 = 0$, the average absolute value of the RoI can be calculated as follows:

$$\langle |r(t)| \rangle = \frac{1}{\pi} \int_0^{\pi} \sin(u)du \; ; \quad u = \frac{2\pi}{T} t \tag{6.4}$$

$$= \frac{2}{\pi}. \tag{6.5}$$

For returns with noise, the average absolute value of $r(t)$ can be calculated by computing the following for a large number of trials N:

$$\langle |r(t)| \rangle = \frac{1}{NT} \sum_{i=0}^{N} \sum_{j=1}^{T} |r(j)| \,. \tag{6.6}$$

Fig. 6.3 shows the numerical calculation of the average absolute value of the RoI for different amplitude and phase noise levels.

Since the distribution of the RoI is independent of the noise level σ_1 for phase noise, it

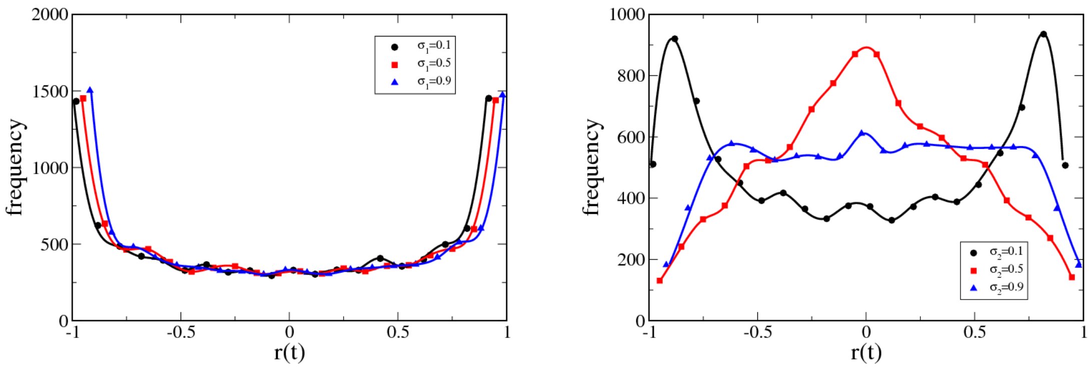


Figure 6.2.: Distribution of the RoI, $r(t)$, Eq. 6.2, for: (left) phase fluctuations $\sigma_1 = \{0.1, 0.5, 0.9\}$ and $\sigma_2 = 0$ and (right) amplitude fluctuations $\sigma_2 = \{0.1, 0.5, 0.9\}$ and $\sigma_1 = 0$. Further parameters: $T = 100$.

is expected that the average absolute RoI is constant with respect to σ_1 in the RoI. This can be seen in Fig. 6.3 (left). On the other hand, as expected from the distribution of the RoI, for the noise level σ_2 for amplitude noise the average absolute RoI varies with respect to σ_2. This can be seen in Fig. 6.3 (right). Roughly, the average absolute value of the RoI with amplitude noise decreases for $\sigma_2 < 0.6$ and it increases for $\sigma_2 \geq 0.6$. This is consistent with the observations for the probability distributions: there, for $\sigma_2 = 0.5$, the values are concentrated around $r(t) = 0$, leading to smaller $\langle |r(t)| \rangle$, and for $\sigma_2 = 0.1$ and $\sigma_2 = 0.9$, the values are less concentrated around $r(t) = 0$, leading to larger $\langle |r(t)| \rangle$.

The correlation between the sign of two consecutive returns shows whether it is possible to draw conclusions from the sign of $r(t)$ based on the sign of $r(t+1)$. In the same manner, numerical calculations were performed to obtain the distribution of the correlations of the RoI with respect to two consecutive returns $r(t)r(t+1)$. These are shown in Fig. 6.4 for both types of noise. We can clearly see that for low levels of noise, there is bigger correlation between consecutive values. As the noise increases, this correlation diminishes until finally, for high levels of noise, the returns are completely uncorrelated.

Most of the algorithms studied are sensitive to correlations in consecutive RoI with the same sign. We notice that between the returns with phase noise and amplitude noise, correlations do not vary exactly in the same manner as noise. In particular, it can be seen that for $\sigma_{1,2} = 0.5$, the amplitude noise still has more correlation than the phase noise. This difference can account for some discrepancies seen between the performance of the agents for the two types of market return functions.

Finally, it is true that returns modeled with a noisy sine wave are not realistic, however, the author believes that they are relevant for any scenario with time series, in which the agent has to choose when and how much to invest.

In the following sections, we specify different strategies for controlling proportion of investment in periodic environments. Afterwards, we compare the performance of these strategies by means of computer simulations. As we mentioned earlier, our primary goal is not to find an optimal strategy for these unrealistic RoIs, but to investigate and analyze the cases in which some strategies perform better than others. Thus, the stylized returns presented in this section are to be considered as a test-bed scenario for different type of investment

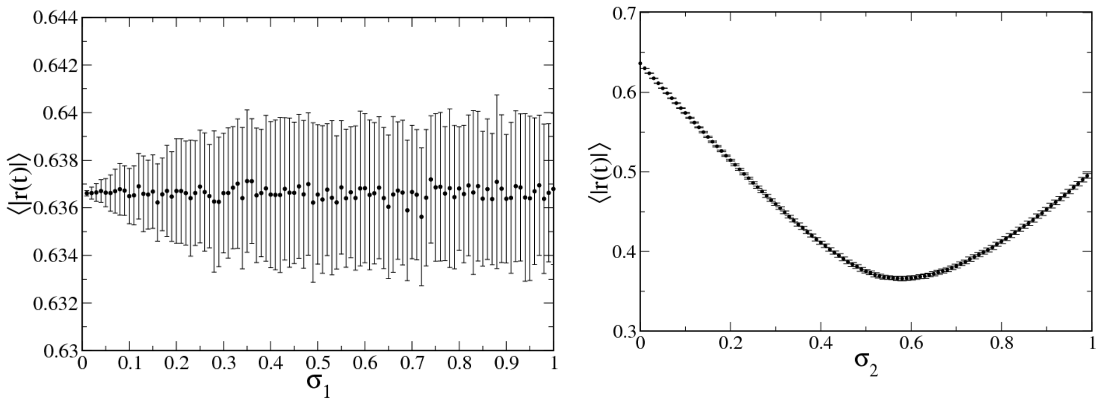


Figure 6.3.: Average absolute value of the RoI, $r(t)$, Eq. 6.2 for: (left) phase fluctuations σ_1 and no amplitude fluctuations, i.e. $\sigma_2 = 0$ and (right) amplitude fluctuations σ_2 and no phase fluctuations, i.e. $\sigma_1 = 0$. Further parameters: $T = 100$.

strategies.

6.4. Reference Strategies

In order to compare different strategies, we need a point of reference against which the performance of each strategy can be measured. For this, we present in this section two strategies that attempt to ease our comparison with other more complex strategies. These reference strategies represent two simple behaviors for an agent; the first one, called *Constant-Investment-Proportion* (CP), assumes a simple constant minimal proportion of investment, whereas the second one, called *Ramp-Rectangle* (RR), increases/decreases the proportion of investment accordingly to the periodicity of the returns. In our approach, the CP strategy represents the agent with zero knowledge whereas the RR strategy represents the agent with complete knowledge of the environment. Note also that in our comparison for the internal complexity of an agent, both reference strategies are considered to be part of a *reactive agent*, as depicted in Fig. 1.2.

6.4.1. Constant Proportion of Investment

The simplest strategy for an agent is to take a constant proportion of investment for every time step. For simplicity we call this strategy CP:

$$q(t) = q_{\min} = \text{const.} \tag{6.7}$$

Since the value of $q(t)$ is always fixed, this is not really a “strategy” and it plays a role in physics-inspired investment models [Navarro-Barrientos et al., 2008b; Solomon and Richmond, 2001b; Sornette and Cont, 1997; Takayasu et al., 1997]. Finally, note that this reference strategy requires **no knowledge** of the dynamics of the RoI.

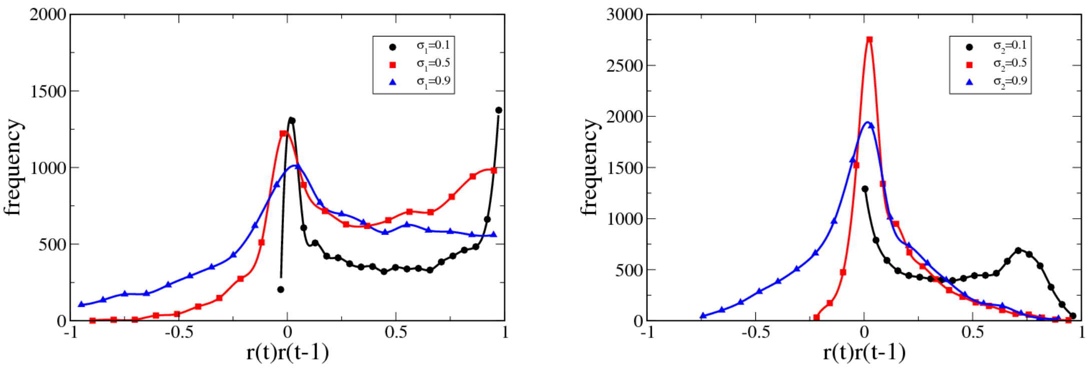


Figure 6.4.: Distribution of the correlations of the RoI over time $r(t)$ Eq. 6.2, for: (left) different phase fluctuations σ_1 with $\sigma_2 = 0$ and (right) different amplitude fluctuations σ_2 with $\sigma_1 = 0$. Further parameters: $T = 100$.

6.4.2. Ramp-Rectangle Strategy

In this section, the case for an agent knowing the dynamics of the periodic returns is considered. Basically, if the agent knows the periodicity T of the returns, then for returns with noise, i.e. $\sigma_1 \neq 0$ or $\sigma_2 \neq 0$, different regions can be observed for which different conclusions can be drawn about the sign of the return. Fig 6.5 (left) shows these regions for some amplitude noise, where in some regions the sign of $r(t)$ is certain to be positive or negative, whereas in other regions the sign of $r(t)$ is uncertain. Based on these regions of certainty and uncertainty, the ramp-rectangle (RR) strategy is proposed to represent the desired behavior of an agent with complete knowledge of the environment.

The ramp-rectangle (RR) strategy is a function mapping RoI that are uncertain to increase/decrease the proportions of investment and RoI that are certainly positive or negative to a maximal or minimal proportion of investment, respectively. The corresponding strategy is expressed as follows:

$$q(t+1) = \begin{cases} \left(\frac{q_{\max}-q_{\min}}{h_1}\right)\hat{t} + q_{\min} & \text{if } \hat{t} \in (0, h_1) \\ q_{\max} & \text{if } \hat{t} \in [h_1, h_2] \\ \left(\frac{q_{\max}-q_{\min}}{h_2-h_3}\right)(\hat{t} - h_2) + q_{\max} & \text{if } \hat{t} \in (h_2, h_3) \\ q_{\min} & \text{if } \hat{t} \in [h_3, h_4]. \end{cases} \tag{6.8}$$

In this function, h_1 (h_3) sets the transition from an increasing (decreasing) ramp function to a rectangle function and $h_2(h_4)$ sets the transition from a rectangle function to a decreasing (increasing) ramp function. Moreover, for each time step t, the congruence $\hat{t} \equiv t \bmod h_4$ is used, which maps each time step $t \in (0, \infty)$ to a time step in the ramp-rectangle function, $\hat{t} \in (0, h_4)$. Fig. 6.5 (right) shows this strategy graphically for a RoI with periodicity $T = h_4$ and no noise.

Furthermore, we assume that the differences between time steps when an agent increases and decreases its proportion of investment values are symmetric. This means that the time difference $\triangle h$ between when the ramp function starts and stops to increase or decrease can

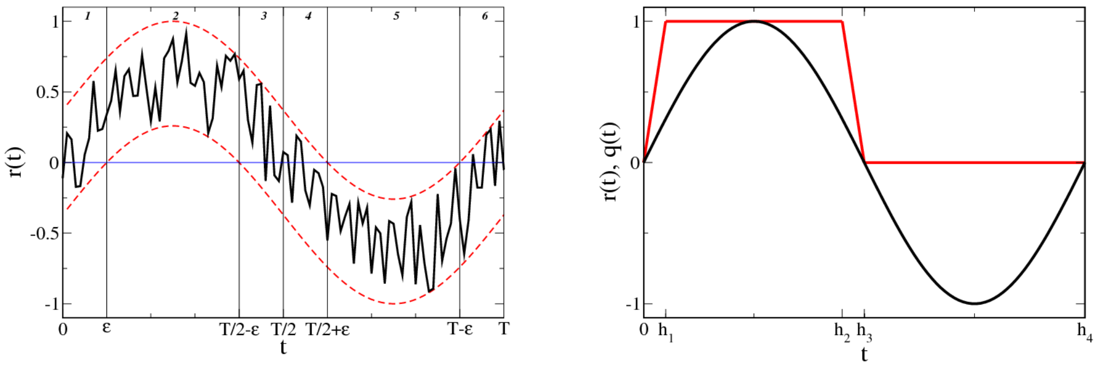


Figure 6.5.: Intervals of certainty and uncertainty: (left) shows $r(t)$ with noise and the different intervals for which different conclusions can be drawn about the sign of the return, where regions 2 and 5 are regions in which the sign of $r(t)$ is certain to be positive or negative, respectively, the rest are regions in which the sign of $r(t)$ is uncertain and (right) shows $r(t)$ without noise and the corresponding $q(t)$ of the RR strategy.

be expressed as follows:

$$\triangle h = h_1 = h_3 - h_2, \tag{6.9}$$

Thus, for the RR strategy, different behaviors for the agent may be proposed. For example, the strategy could be defined in such a way that the proportion of investment should increase only for the time steps where returns are **certain to be positive** and not for the whole positive period of the returns. For returns with amplitude noise, this would mean that the agent wants to avoid any loses and takes always into account the worst possible outcomes which, in terms of the returns, can be expressed as follows:

$$r_w(t) = (1 - \sigma_2) \sin\left(\frac{2\pi}{T}\, t\right) - \sigma_2. \tag{6.10}$$

It can be shown that by solving $r_w(t) = 0$ for t, the time step intervals in a cycle for which the returns are certain to be positive is:

$$[\varepsilon, (T/2) - \varepsilon]\,, \tag{6.11}$$

where:

$$\varepsilon = -\frac{T}{2\pi} \arcsin\left(\frac{\sigma_2}{\sigma_2 - 1}\right). \tag{6.12}$$

This interval corresponds to region 2 in Fig. 6.5 (left), where large investments should be performed as the returns are certain to be positive.

It is important to notice that this **reference strategy** assumes that the agent **knows** the **periodicity** and the **noise** of the returns in advance.

A more daring behavior would be if the agent takes only the periodicity into account and forgets about the noise and the ramp-function, i.e. $\triangle h = 1$ in Eq. 6.9, in which case the agent uses a *Square Wave* (SW) strategy. We are particularly interested in this case in which it is implied that an agent invests q_{max} for time steps $\hat{t} \in (0, T/2)$ and invests q_{min} for time steps $\hat{t} \in [T/2, T]$. For the sake of completeness, this strategy is expressed as follows:

$$q(t) = \begin{cases} q_{\max} & t \bmod T < T/2 \\ q_{\min} & \text{otherwise.} \end{cases} \tag{6.13}$$

Note that this **reference strategy** assumes that the agent **knows** the **periodicity** of the returns in advance.

6.5. Strategies based on Technical Analysis

For the investment strategies presented in this section, we assume that a strategy consists of two components: a *prediction component* and an *action component*. For these strategies, the prediction component predicts a variable in the system – in this case, the next value of $r(t)$ – and the action component defines an action based on the prediction of the variable. In this case, it defines the appropriate value for $q(t)$.

For the strategies presented in this section, we assume that the adjustment of the proportion of investment is based on "technical analysis" (see [Brooks, 2002; Stock, 1993]). Technical analysis tries to deduce information about the dynamics of, for example, stock market prices by looking at trends (averages, variances, higher order moments) of the values over a period of time. This assumes that an agent has a bounded memory of size M to record previous prices or RoIs; this information is then processed in different ways in order to predict the next value. In our approach we do not consider prices but returns which are more appropriate for our investment model expressed in terms of returns on investment.

In the following, we consider two strategies from the field of technical analysis: the first strategy calculates the *moving averages* (MA) on previous RoI, while the second strategy uses *moving least squares* (MLS) on previous RoI $r(t)$ over a fixed period of time M. Both of them can be regarded as "zero-intelligence" strategies, as agents do not do any reasoning or learning. Note also that in our comparison for the internal complexity of an agent, these types of strategies are mean to be part of the internal architecture of an *experience-based agent*, depicted in Fig. 1.2.

6.5.1. Moving Averages

The moving averages technique estimates the next $r(t)$ as the average of the previous M values of $r(t)$:

$$\hat{r}_{MA}(t) = \frac{1}{M} \sum_{n=t-M}^{t-1} r(n). \tag{6.14}$$

6.5.2. Moving Least Squares

The moving least squares technique fits a function to the data of the previous M values of $r(t)$ to estimate the next $r(t)$. In our case, we choose this function to be a linear trend-line, which is found by minimizing the distance to the data points of $r(t)$. Based on the previous M values of $r(t)$, the squared estimation error ϵ_r is defined as:

$$\epsilon_r(t) = \frac{1}{M} \sum_{n=t-M+1}^{t} \left[r(n) - \hat{r}_{MLS}(n)\right]^2 \tag{6.15}$$

where $\hat{r}(t)$ is the predicted RoI based on the linear regression trend-line, defined as:

$$\hat{r}_{MLS}(t') = m(t)\, t' + b(t) \quad \text{for } t - M \leq t' \leq t. \tag{6.16}$$

Now, the best fitting values m and b are obtained by minimizing the squared error estimation, Eq. (6.15). From $\partial\epsilon_r/\partial m = 0$ and $\partial\epsilon_r/\partial b = 0$, we get:

$$m(t) = \frac{M \sum_{n=t-M+1}^{t} n\, r(n) \; - \; \left(\sum_{n=t-M+1}^{t} n\right)\left(\sum_{n=t-M+1}^{t} r(n)\right)}{M \sum_{n=t-M+1}^{t} n^2 - \left(\sum_{n=t-M+1}^{t} n\right)^2} \tag{6.17}$$

$$b(t) = \frac{1}{M}\left[\sum_{n=t-M+1}^{t} r(n) - m(t) \sum_{n=t-M+1}^{t} n\right] \tag{6.18}$$

Fig. 6.6 shows the values estimated for $r(t)$ when using MA and MLS for a sample scenario over 100 time steps and with parameters: $M = 20$ and $\sigma_1 = 0.5$. It can be observed that the strategy *MLS* estimates $r(t)$ more precisely than *MA*. For *MA*, the values of the estimation have a 'lag' as compared to the actual $r(t)$ which may cause the actual $r(t)$ to be underestimated/overestimated for increasing/decreasing $r(t)$. For *MLS*, the values of the estimation do not present a 'lag' when compared to the actual $r(t)$ – this is not surprising as the curve of the estimation is fitted to minimize the distance between the values of the estimation and the actual $r(t)$.

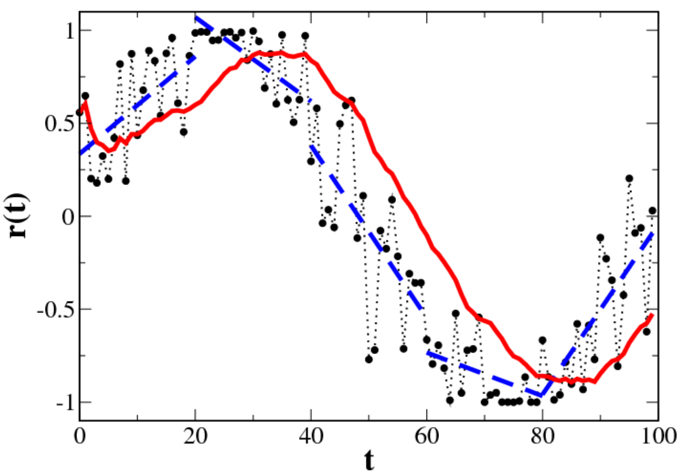


Figure 6.6.: Moving averages (MA, solid line) and moving least squares (MLS, long-dashed line, shown for $t = 20, 40, 60, 80, 100$) for the time series of a periodic return on investment $r(t)$ (dot points with short-dashed line) with some noise. Parameters: $M = 20$, $\xi = N(0, 0.5)$.

6.5.3. Daring and Cautious behaviors

The strategies MA and MLS use different approaches to estimate the next $r(t)$. However the corresponding adjustment of the proportion of investment must still be defined. For this, we consider two possibilities: a *daring* (RS) and a *cautious* (RA) approach.

For a daring approach, the value of $q_{RS}(t)$ is defined as follows for $\hat{r}(t) \in \{\hat{r}_{MA}(t), \hat{r}_{MLS}(t)\}$, i.e. for $\hat{r}(t)$ being an MA or MLS estimate of $r(t)$:

$$q_{RS}(t) = \begin{cases} q_{\min} & \hat{r}(t) \leq 0 \\ q_{\max} & \hat{r}(t) > 0 \end{cases} \tag{6.19}$$

where $q_{\min}, q_{\max} \in [0, 1]$ and $q_{\min} < q_{\max}$. In other words, the agent invest the percentage $q_{\min}$ of its budget if the next value of $r(t)$ is predicted to be negative or zero, and the agent invests the percentage $q_{\min}$ of its budget if the next value of $r(t)$ is predicted to be positive.

For a cautious approach, the value of $q_{RA}(t)$ is defined as follows for $\hat{r}(t) \in \{\hat{r}_{MA}(t), \hat{r}_{MLS}(t)\}$, i.e. for $\hat{r}(t)$ being an *MA* or *MLS* estimate of $r(t)$:

$$q_{RA}(t) = \begin{cases} q_{\min} & \hat{r}(t) \leq q_{\min} \\ \hat{r}(t) & q_{\min} < \hat{r}(t) < q_{\max} \\ q_{\max} & \hat{r}(t) \geq q_{\max} \end{cases} \tag{6.20}$$

where $q_{\min}, q_{\max} \in [0, 1]$ and $q_{\min} < q_{\max}$. Here, $q_{RA}(t)$ is set to the predicted $r(t)$ (with appropriate adjustments to ensure that $q_{RA}(t) = q_{min}$ whenever $\hat{r}(t) \leq q_{min}$ and $q_{RA}(t) = q_{max}$ whenever $\hat{r}(t) \geq q_{max}$) – the agent only invests a fraction of the budget which corresponds in size to the expected return.

Note that in terms of economic theory and in particular in terms of risk theory, the behavior of the agent is *risk-neutral* in the sense that the agent estimates only the expected return $\hat{r}(t)$ and does not consider risk measures such as the volatility. Thus, this approach is based on the estimation of $r(t)$, and given this estimation, the agent makes the decision to increase or decrease the proportion of investment. Hence, for the purposes of this thesis, the two terms 'daring' and 'cautious' denote only the investment preference of the agent in terms of expected return. See more details of this approach with respect to utility of wealth in the Appendix 10.2.

6.6. Strategies based on Machine Learning

In the previous sections, we presented two reference strategies for reactive agents and two strategies based on technical analysis methods for experience-based agents. Now, in this section, we present other types of strategies which are based on methods taken from the field of machine learning. We consider two different approaches: one based on an incremental update rule (IUR), which is a form of reinforcement learning, and the other based on a genetic algorithm (GA), which is a form of evolutionary learning. Note also that in our comparison of the internal complexity of an agent, these types of strategies are mean to be used by *machine learning-based agents* with an internal architecture as depicted in Fig. 1.2.

6.6.1. Incremental Update Rule

The following machine learning approach is based on the incremental update rule, an application of reinforcement learning. The idea of reinforcement learning is that an agent continuously uses a reward signal to adjust its own performance. In our scenario, the return is the reward signal; at each step, the agent computes the error between the predicted and the actual value of the return and uses this error to adjust the estimation of the following return. The general incremental update rule from reinforcement learning is defined as follows [Sutton and Barto, 1998]:

$$NewEst \leftarrow OldEst + StepSize[\,Target - OldEst\,] \tag{6.21}$$

where $OldEst$ and $NewEst$ are the old and new estimates for the quantity of interest. So, $Target - OldEst$ gives us the error of the current estimation, which is weighted by the factor $StepSize$. Thus, a new estimate $NewEst$ is computed by taking the old estimate and adjusting it by the error of the current estimate, leading to a $NewEst$ that is updated at each time step. Applying Eq. (6.21) to our model, we find the following instance of the incremental update rule:

$$\hat{r}_{IUR}(t+1) = \hat{r}_{IUR}(t) + \gamma[r(t) - \hat{r}_{IUR}(t)] \tag{6.22}$$

Consequently, $OldEst$ and $NewEst$ are the old and new estimates for the return $\hat{r}_{IUR}(t)$ and $\hat{r}_{IUR}(t+1)$ respectively. Furthermore, $r(t) - \hat{r}_{IUR}(t)$ is the error of the current estimate. Because of its recursive definition, the incremental update rule considers an infinite history of returns – of course, the weight of a value depends on its age and its impact fades over time. For simplicity, we chose $\hat{r}_{IUR}(0) = 0$ as the initial value of $\hat{r}_{IUR}(t)$.

Note that different values of γ lead to different performance for the algorithm; in other words, for small γ, the adjustment of the estimate will be small and for large γ, the adjustment of the estimate will be large. This is investigated in Section 7.2.3, where we discuss the γ values that yield maximal gains in more detail.

Finally, as in Section 6.5.3, we must specify the corresponding adjustment of the proportion of investment given a particular estimate for the next return. For the purpose of comparison, we again define a daring and a cautious approach for the MA and MLS strategies as in Eq. 6.19 and 6.20:

In the daring approach, $q_{RS}(t)$ is defined as follows for $\hat{r}_{IUR}(t)$:

$$q_{RS}(t) = \begin{cases} q_{\min} & \hat{r}_{IUR}(t) \leq 0 \\ q_{\max} & \hat{r}_{IUR}(t) > 0 \end{cases} \tag{6.23}$$

and in the cautious approach, $q_{RA}(t)$ is defined as follows for $\hat{r}_{IUR}(t)$:

$$q_{RA}(t) = \begin{cases} q_{\min} & \hat{r}_{IUR}(t) \leq q_{\min} \\ \hat{r}_{IUR}(t) & q_{\min} < \hat{r}_{IUR}(t) < q_{\max} \\ q_{\max} & \hat{r}_{IUR}(t) \geq q_{\max}. \end{cases} \tag{6.24}$$

For both definitions, $q_{\min}, q_{\max} \in [0, 1]$ and $q_{\min} < q_{\max}$.

It is important to note that reinforcement learning and the incremental update rule are not identical; rather, reinforcement learning describes a group of machine learning approaches and the incremental update rule is one instance of these approaches.

It is well known that people make decisions based on changes from a certain reference point, that people react differently when decision-making for profits and for losses and that people perceive losses to be twice as large as profits [Edwards, 1996; Kahneman and Tversky, 1979, 1992; Takahashi and Terano, 2003]. Although these concepts are not explicity considered here, a different representation of γ in Eq. (6.22) could be used to study some aspects of the Prospect Theory of decision-making.

6.6.2. Genetic Algorithm

In this section we present an investment strategy based on a Genetic Algorithm *(GA)*. Genetic algorithms (GA) are a technique from the field of artificial intelligence which is mainly used to find approximate solutions to optimization problems. Genetic algorithms belong to the class of evolutionary algorithms that are based on the principle of modeling solutions to a problem by means of a population of chromosomes where each chromosome represents a candidate solution to the problem and where the population gradually evolves to better solutions via operators like selection, crossover and mutation.

In the following, a description of the genetic algorithm for controlling the proportion of investment is presented [Navarro-Barrientos et al., 2008a]. Let $j = 1, ..., C$ be a chromosome from a population of chromosomes of size C. Each chromosome j is an array of genes g_{jk} ($k = 0, ..., G-1$) where the values of the genes are real numbers (see [Michalewiçz, 1999, chap. 5] for drawbacks when using binary representation in genetic algorithms). In our model, each chromosome j represents a *set of possible strategies* of an agent, so the g_{jk} refers to possible proportion of investment values.

In the beginning, each g_{jk} is assigned a random value: $g_{jk} \in (q_{min}, q_{max})$. Each chromosome j is then evaluated by a *fitness function*, $f_j(\tau)$, which is defined as follows:

$$f_j(\tau) = \sum_{k=0}^{G-1} r(t)\, g_{jk}\ ; \quad k \equiv t \bmod G. \tag{6.25}$$

In our model, the fitness is determined by the gain/loss that each strategy g_{jk} yields depending on the RoI $r(t)$. Since the fitness of a chromosome must to be maximized, negative $r(t)$ lead to very small values of g_{jk}, i.e. a low proportion of investment, whereas positive $r(t)$ lead to larger values of g_{jk}. This lets us consider the product of $r(t)g_{jk}$ as a performance measure of a chromosome in accordance with the budget dynamics of Eq. (6.1).

The values of g_{jk} are always multiplied by different $r(t)$ values i.e. depending on t. For the chromosome, we define a further time scale τ in terms of generations. A generation is completed after each g_{jk} is multiplied by an RoI from consecutive time steps t. This means that the index k refers to a particular time t in the following manner: $k \equiv t \bmod G$, which means $k = \hat{t} \in \{1, G\}$, with $t = \hat{t} + \tau G$, $\tau = 0, 1, 2, ...$

After time τ, the population of chromosomes is replaced by a new population of better fitting chromosomes with the same population size C. This new population is selected from the previous population in the following manner: after calculating the fitness of each chromosome according to Eq. (6.25), we find the best chromosomes from the old population by applying elitist and tournament selection of size two:

- *Elitist selection* considers the best *percentage* s of the population which is found by ranking the chromosomes according to their fitness. The best chromosomes are directly transferred to the new population.

- *Tournament selection* is done by randomly choosing two pairs of two chromosomes from the old population and then selecting from each pair the one with the higher fitness. These two chromosomes are not simply transfered to the new population, but undergo a transformation based on the genetic operators crossover and mutation, as follows: the single-point crossover operator finds the cross point, or cut point, in the two chromosomes beyond which the genetic material from two parents is exchanged to form two new chromosomes. This cut point is the integer part of a random number drawn from a uniform distribution $p_c \in U(1, G)$.

After the crossover, a mutation operator is applied to each gene of the newly formed chromosomes. With a given mutation probability $p_m \in U(0, 1)$, a gene is to be mutated by replacing its value with a random number from a uniform distribution $U(q_{min}, q_{max})$. After the cycle of selection, crossover and mutation is completed, we arrive at a new population of chromosomes that consists of a percentage of the best fitted chromosomes from the old population plus a number of new chromosomes that ensure further possibilities for the evolution of the set of strategies.

Given the optimized population of chromosomes representing a set of possible strategies, the agent still needs to update its actual proportion of investment $q(t)$. This is done as follows: at time $t = \tau$, the agent takes the set of strategies g_{jk} from the chromosome j with the highest fitness in the previous generation. Given $G = T$, this means that the agent for each time step of the upcoming cyclic change chooses the appropriate proportion of investment by computing the following:

$$q_{GA}(t) = g_{jk} \quad \text{with } j = \arg\left(\max_{j=1,\ldots,C} f_j\right) \; ; \quad k \equiv t \bmod G. \tag{6.26}$$

In the following section we will adjust the respective parameters of each of these strategies so that they lead to maximal gains. This is crucial if we want to compare the performance of these different investment strategies – because a fair comparison can be only assured if the strategies are performing at their best.

6.7. Summary

In this chapter, different investment strategies were presented that can be applied by agents in an investment market scenario with periodic returns and different types and levels of noise. Note that for the strategies presented in this chapter, three different levels of information available for the agent were considered: no knowledge (CP strategy), partial knowledge (MA, MLS, IUR and GA strategies) and complete knowledge (RR and SW strategies). For simplicity, according to the level of available information, different type of investment strategies have been chosen from simple approaches like constant proportion of investment, technical analysis and evolutionary approaches. In this PhD thesis, a risk-neutral approach is considered for the strategies; however, note that in economics, the risk-averse and risk-seeking approaches are more revised and the decision to increase or decrease the amount of investment depends on the variance of the returns. For further details, see some contributions on behavior towards risk [Arrow, 1965; Hens and Schenk-Hoppé, 2006; Hnilica, 2002; Pratt, 1964; Tobin, 1958].

On the other hand, note that different investment strategies may also be proposed based on different types of available information. For example, it is usually assumed that there exist two types of investors in the stock market:

- Fundamentalist - consider the fundamental values of a company and believe that all market moves are driven by basic changes within the industry or world conditions.
- Technicians - also called chartists, use technical tools to predict the movement of prices.

In this chapter, the methods of *Moving Averages* Section 6.5.1 and *Moving Least Squares* Section 6.5.2, were considered to find the tendency of the returns. However, other more complex methods using *Moving Averages* are used to find the tendency of the returns. For example, if we consider Moving Averages with different window sizes, whenever a Moving Average is traversed, this may indicate a change in the tendency. Chartists also use different types of market patterns that predict changes in the tendency of the prices, for example the use of *support* and *resistance* lines. When the price reaches the support(resistance) boundary, the market (which may be understood like the will of thousand of investors) considers that the price is too low/high and more investors may be willing to start buying/selling their stocks. Moreover, there are also different figures which are used by chartists to predict asset prices, for example the double-peak and double-valley figures are two typical patterns for predicting changes in tendency. Another example is the pattern head-and-shoulders pattern which anticipates a decay of the asset price. Extra information like volume is also useful for chartists. Here, a decay in volume together with a decay in the price confirms the decaying tendency.

Finally, it would be also interesting to include investment strategies based on *Value Functions* from Prospect Theory [Kahneman and Tversky, 1979], which are based, as mentioned earlier, on the fact that people tend to perceive losses to be twice as large as profits (see [Edwards, 1996; Kahneman and Tversky, 1992; Takahashi and Terano, 2003]). However, all these different approaches that we briefly mentioned are beyond the scope of this thesis and are left for further work.

7. Comparison of Investment Strategies for Noisy Periodic Environments

This chapter compares the performance of different agent strategies. The performance is measured by the average budget growth obtained after a certain number of time steps. Results are presented for extensive computer simulations, in which it is shown that for exogenous returns with periodicity: (i) a daring behavior outperforms the cautious behavior, and (ii) the genetic algorithm is able to find the optimal investment strategy by itself, thus outperforming the other strategies considered.

7.1. Introduction

In the previous chapter, different investment strategies for noisy periodic environments were presented. Now, in this chapter, the performance of the different strategies is compared and analyzed.

This chapter is organized as follows. Firstly, Section 7.2 shows the optimal parameter adjustment for the strategies presented in the previous chapter. And secondly, in Section 7.3 the performance of the strategies for different types and levels of noise is compared and analyzed.

7.2. Parameter Tuning

In the previous chapter, different strategies were presented. These strategies can be used by an agent to adjust its investment proportion $q(t)$, i.e. to determine two important tasks: when and how much money to invest in the market. As mentioned earlier, the main goal in this chapter is to compare the performance of these strategies in a periodic environment. However, in order to make this comparison meaningful, we have to ensure that we have adjusted the different parameters of the strategies properly. Only if the strategies perform at their optimum, can they really be compared. Thus, in this section, we deal with the problem of finding the parameter values that lead to maximal gains for each strategy.

Furthermore, in order to find which are the optimal parameter values for each strategy, we need to define what we mean by strategy optimality. For this, we choose to measure the performance of agents as the average of their budget growth over a certain number of time steps. The optimal strategy is the strategy that performs better than all the other strategies, i.e. the strategy that, on average, leads to the greatest budget growth.

More precisely, we assume that a strategy is optimal in a given time interval, $\{t_{\text{start}}, t_{\text{end}}\}$, if it leads to the *maximum total budget* during that interval, where a large value is chosen for t_{end}, e.g. $t_{\text{end}} = 10^4$ time steps in most of the experiments, to avoid intermediate effects. The starting value t_{start} has to be chosen in such a way that each agent has enough time to gather the information necessary for the proper calibration of the algorithm that it applies.

Therefore, we have set $t_{\text{start}} = T$ in order to ensure that the agent will have gathered at least a cycle of the RoI.

When evaluating the strategies, we have to consider that their performance is also influenced by stochastic effects because of the RoI, Eq. (6.2). This means that we have to average the simulation over a large number of trials $N_a = 10^3$ where each trial simulates an agent acting independently with the same set of strategy parameters. More specifically, the performance of an agent in a single trial corresponds to the average budget at the end of each RoI's period, T; afterwards, an average is taken over a number of trials to diminish noise effects. Mathematically this can be expressed as follows:

$$\langle x \rangle = \frac{1}{N_a} \sum_{i=1}^{N_a} \frac{1}{I} \sum_{k=0}^{I} x(k\,T) \; ; \quad I = \frac{t_{\text{end}} - t_{\text{start}}}{T}. \tag{7.1}$$

For convenience, in Eq. (7.1), the total budget has been normalized by the number of cycles or periods of the RoI, I. This is done for the following three reasons. In the first place, for a constant investment action and a return function with no noise, this average value will have zero standard deviation. In contrast, if we take the average growth over all of the time steps, there will be a non-zero standard deviation associated with the sine wave. Secondly, if the agent's strategy is able to forecast the next RoI correctly, then the agent's budget may reach very high values in a very short time. This can be seen in the dynamics of Eq. (6.1), where the budget can be doubled at each time step if an appropriate $q(t)$ and $r(t)$ are provided. In the computer simulations this would lead to numerical overflows. Therefore, we have chosen to reinitialize the budget after each cycle of the RoI, i.e. $x(k\,T) = x(0)\;; k = 1, 1, .., I$, (which applies to all simulations, to ensure comparison). Thirdly, if returns have noise on the phase and the performance is measured at the end of each cycle, it can be shown that the initial phase value does not modify the average budget obtained at the end of a cycle (see Appendix 10.2).

Thus, the procedure that we apply to find the optimal parameters is straightforward: we compare the performance – averaged over 10^3 periods – of each of the algorithms for a range of possible parameters and then choose the optimal one.

Moreover, we first consider the case where the agent adjusts its proportion of investment according to a cautious behavior, *RA* (for simplicity also called risk-avoiding behavior), and we assume that the agent receives RoI with no *phase* noise $\sigma_1 = 0$ for different *amplitude* noise levels σ_2.

7.2.1. For Reference Strategies

For the reference strategy *CP*, Eq. (6.7), there is no need to optimize any parameters because as it was shown in Section 3.2, small constant proportion of investments lead to larger profits in scenarios with random returns. Thus, an agent using the strategy *CP* has the minimum allowed constant proportion of investment $q = q_{\min} = 0.1$. For the reference strategy *RR*, Eq. (6.8), however, we note that we are required to find the best $\triangle h$, Eq. (6.9), for every noise level. Recalling Fig. 6.5 (right), the interval for which returns are certain to be positive can be obtained by calculating ε for the given noise levels and the periodicity of the returns using Eq. 6.12. Fig.7.1 (left) shows, for returns with periodicity $T = 100$ and different amplitude noise σ_2 the ε values obtained using Eq. 6.11. This is used to determine the interval in which returns are certain to be positive. For example, for $\sigma_2 = 0.1$ and no noise on the phase $\sigma_1 = 0$, it can be seen that $\varepsilon = 1.77$ leads to the interval $\hat{t} \in [2, 48]$, in which returns are certain to be positive. Note that, for $\sigma_1 + \sigma_2 \geq 0.5$, this leads to an $\varepsilon \geq T/4$, which means that there is no interval in which returns are certain to be positive.

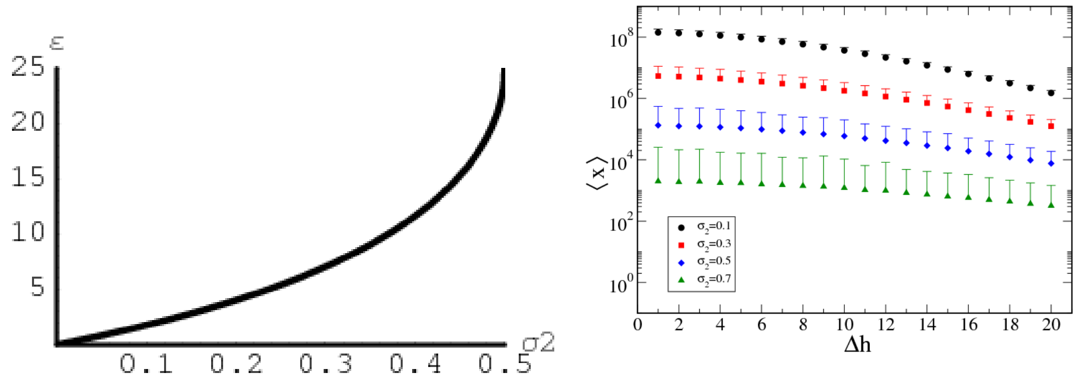


Figure 7.1.: For returns with periodicity $T = 100$ and different amplitude noise levels σ_2: (left) different ε values, Eq. (6.12), used to find the interval where returns are certain to be positive; and (right) average budget obtained using strategy RR, Eq. (6.8) for different $\triangle h$ values, used to determine the proper $q(t)$, Eq. (6.9).

Now, in order to elucidate the performance of these previous approaches for the strategies RR, Eq. (6.8), and SW, Eq. (6.13), consider an agent with initial budget $x(0) = 10$, the wealth dynamics as described in Eq. (6.1) and the following parameters for both strategies: $q_{\min} = 0.1$ and $q_{\max} = 1.0$.

If the agent uses strategy SW, we find that for returns with no noise, $r(t) = \sin(2\pi t/T)$, $T = 100$ and that after $t = 100$ time steps the budget of the agent would be $x(100) = 6.746 \times 10^9$. Now, assuming that returns have some amplitude noise, $\sigma_2 = 0.1$, and periodicity $T = 100$, it can be seen that after $t = 100$ time steps, in the worst case, Eq. (6.10), the strategy SW leads to the budget $x(100) = 1.489 \times 10^9$.

On the other hand, if the agent uses the strategy RR and maps the time difference $\triangle h$ with the intervals where returns are certain to be positive, Eq. (6.11), as follows:

$$\triangle h = \varepsilon, \tag{7.2}$$

then, it can be shown that for $T = 100$ and $\sigma = 0.1$, returns are certain to be positive for time steps in the range $[2, 48]$ and at the end of a cycle, this strategy would lead to a budget of $x(100) = 1.226 \times 10^9$. Note that the latter is less than the budget that can be obtained using the strategy SW for the same scenario. Furthermore, for $\sigma_2 = 0.5$, if (despite the noise) the agent decides to use the strategy SW, this leads after $t = 100$ time steps to the budget $x(100) = 1.373 \times 10^6$. It can be shown that for an agent using the strategy RR and deciding to invest all its budget in the intervals $[2, 48], [10, 40], [24, 26]\}$, these lead to the budgets $x(100) = \{1.23 \times 10^6, 116015, 13.41\}$, respectively. This means that even for returns with large levels of noise, from these examples we may think that the best strategy is to increase the proportion of investment once the returns are more likely to be positive than negative (SW strategy) and not only for the returns that are certain to be positive (RR strategy using Eq. (7.2)). These statement is corroborated in Fig. 7.1 (right), where we show the average budget obtained from some simulations of the dynamics for different

$\triangle h$ values and different amplitude noise levels. Thus, based on visual impression, we now find by means of simulations that the best parameter value for the strategy RR is $\triangle h = 1$, i.e. the strategy SW, Eq. (6.13). Curiously, even for large noise levels, if the agent wants to increase his profits it is much better to increase/decrease the proportion of investment than slowly. Recalling that the goal of the agents in our simulations is to maximize the budget and not to minimize the loss, then it may be clear that even with large fluctuations it is better for the agents to bet for a win than for a loss.

Note that for the reasons presented previously, we include the strategy SW and exclude the strategy RR from our performance comparison in Section 7.3.

7.2.2. For Strategies Based on Technical Analysis

The main parameter in the two investment strategies presented in Section 6.5 is the memory size M. This parameter plays an important role on the estimation of returns, for example if the memory is large, there would be cases for which the probability that current returns are positive is large, but given that there are still negative returns in the memory of the agent, the agent may estimate a negative return yielding a decrease in the proportion of investment. On the other hand, it may occur that current returns are negative, but given the large memory, the agent is still accounting for previous positive returns, which may lead the agent to estimate a positive return and to increase the proportion of investment; consequently the agent may turn a loss. In other words, a large memory is not always advantageous because the agent may still remember positive returns even if the current cycle has large negative returns and vice versa.

For Strategy Using *Moving Averages*

We begin our analysis by showing the proportion of investment for an agent using the MA strategy, see Eq. (6.14), for different memory size values. Fig. 7.2 (left, bottom) shows the evolution of the proportion of investment $q(t)$ for different M values for returns with no noise Fig. 7.2 (left, top).

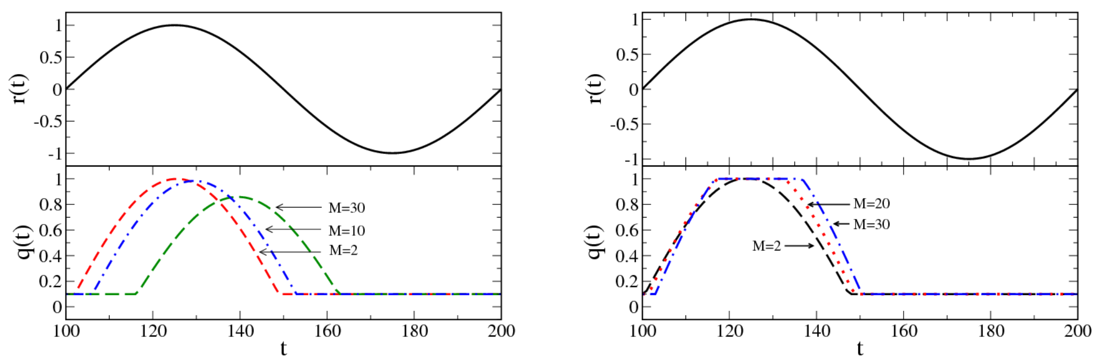


Figure 7.2.: (above) Periodic returns with $T = 100$ and no noise and (bottom) proportion of investment values, both using (left) MA and (right) MLS strategies for different memory size M.

It is clear, that for small memory sizes, the correspondence between proportion of investment values and return is much better than for large memory sizes, for which the lag is also large. It is also important to note that the amplitude of the proportion of investment value decreases if the memory is large. However, if returns have noise, performing an average with a large memory may help to account for the fluctuations in the returns and to allow agents to avoid investing too much when estimations are negative.

Thus, in order to find the proper memory size M, we first assumed RoIs with different amplitude fluctuations, $\sigma_2 = 0.1, 0.3, 0.5, 0.7$, and no phase fluctuations, $\sigma_1 = 0$. Afterwards, using the *Monte Carlo Simulation* approach in Eq. (7.1), some experiments for an agent using a moving average strategy were performed for different memory size values and for different noise values.

Fig. 7.3 (left) shows the average of the budget of the agent using strategy *MA* for different memory size values and for different amplitude noise level σ_2. It is clear that the performance is much better for smaller memory sizes than for larger memory sizes. Thus, based on visual impression, a proper memory size value for the strategy *MA* is $M = 2$.

Furthermore, for the scenario with no amplitude noise, $\sigma_2 = 0$ and different phase fluctuations, the best memory size value was $M = 5$.

For Strategy Using *Moving Least Squares*

As for the *MA* strategy, Fig. 7.2 (right, bottom) shows the different proportion of investment values using strategy *MLS*, see Eq. (6.14), for different memory size values. As expected, for small memory size values, both strategies *MLS* and *MA* behave almost the same. Also note that for both strategies, the larger the memory size, the larger the lag over time. However, we note that for some memory size values, the intervals with maximal proportion of investment values using *MLS* are larger than for the *MA* strategy, which may lead to higher profits for small noise levels. This occurs because for positive increasing returns, large memory sizes may lead to return estimates much larger than $\hat{r} = 1$, due to the fact that the proportion of investment is constrained to be in the range $q \in (0, 1)$, see Eq. (6.20). This saturation effect on q values occurs for large memory sizes, see Fig. 7.2 (right, bottom).

Again, we investigate the best memory size value that should be used for this strategy, using Eq. (7.1). Fig. 7.3 (right) shows the average of the agent's budget for different memory size values and for different amplitude noise levels σ_2.

Based on visual impression, the memory size of $M \approx 37$, leads to the highest profits when using the strategy *MLS*. It can be seen that the strategy *MLS* leads to higher profits than the *MA* strategy. This occurs because of the saturation effect discussed previously, see Fig. 7.2 (right, bottom).

Also for the scenario with no amplitude noise, $\sigma_2 = 0$ and different phase fluctuations, it was found empirically that $M \approx 37$ was the best memory size value. Furthermore, some experiments were performed for different periodicity T and no noise and it was found experimentally that when using the strategy *MLS*, the relationship between the best memory size M and the periodicity T are proportional; $M/T \approx 0.37$. This is shown in Fig. 7.4 for returns with different T with no amplitude and no phase noise.

Moreover, if no noise is assumed, the problem is analytically tractable. For this, we note that for a periodic return as in Eq. (6.2) with parameters $A = 1$ and $\sigma_1 = \sigma_2 = 0$, the *MLS*

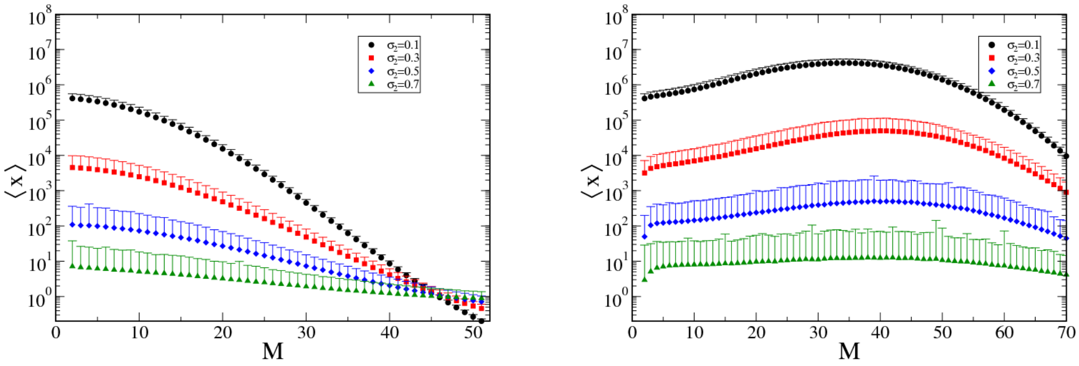


Figure 7.3.: Average of the agent's budget for strategies: (left) *MA* and (right) *MLS*; for different memory sizes and different amplitude noise levels. Parameters: $T = 100$, $\sigma_1 = 0$, $t = 10^4$, $N_a = 10^3$.

strategy estimates the next return $\hat{r}(t+1)$ as follows:

$$\hat{r}(t+1) = \frac{\sin(\omega\, t) - \sin(\omega\,(t-M))}{\omega\, t - \omega\,(t-M)} \tag{7.3}$$

$$= \frac{M+1}{M}\left[\sin(\omega\, t) - \sin(\omega\ (t-M))\right], \tag{7.4}$$

where $\omega = \frac{2\pi}{T}$. Now, by calculating the average profits $\langle r\, q\rangle$ for the positive cycle of the returns for q as in Eq. (6.20), we find:

$$\langle r\, q\rangle = \int_0^{T/2} r(t)\, q(t) dt \tag{7.5}$$

$$= \frac{M+1}{M}\int_0^{T/2}\left[\sin(\omega\, t)\left(\sin(\omega\, t) - \sin(\omega\ (t-M))\right)\right] dt \tag{7.6}$$

$$= \frac{T\,(M+1-\cos(\omega\, M))}{4\, M}. \tag{7.7}$$

Fig. 7.4 (right) shows the resulting memory size values for Eq. 7.5. Note that the memory size that leads to maximum profits can be found by derivation of $\langle r\, q\rangle$ w.r.t M, which leads to:

$$\partial_M \langle r(t)\, q(t)\rangle = \frac{-T\sin\left(\frac{\omega}{2}\, M\right)^2 + \pi\, M\, \sin(\omega\, M)}{2\, M^2}. \tag{7.8}$$

The memory size, $M^\star$, that maximizes the profits can be calculated by solving $\partial_M \langle r(t)\, q(t)\rangle = 0$ for M, which using Taylor series to the 6th order for the sinusoidal functions leads to the following expression [Navarro-Barrientos, 2008b]:

$$M^\star = \frac{\sqrt{\frac{3}{2}}}{\pi}\, T. \tag{7.9}$$

Consequently, for $T = 100$, the theoretical optimal memory size is $M^\star \approx 38$, which is pretty near to the previous empirical result shown in Fig. 7.3 (right). Interestingly, the proportion

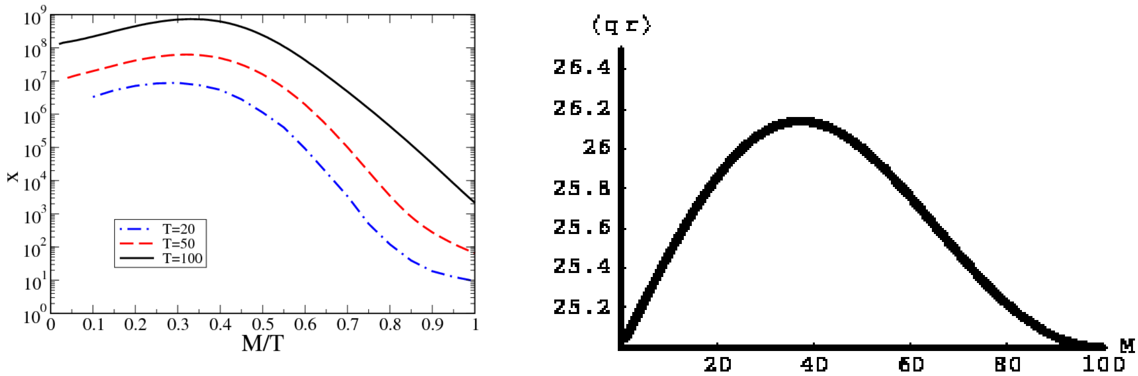


Figure 7.4.: For strategy *MLS*, both plots show for different memory size M and no noise: (left) agent's budget for returns with different periodicity T and (right) analytical calculation of the average of profits. Note that the largest $\langle x \rangle$ and $\langle r\, q \rangle$ is reached for the memory size values as in Eq. (7.9).

$M/T \approx 0.37$ was found by means of computer simulations (see Fig. 7.4 (left)), in which it can be seen that the best memory sizes in the simulations pretty well approximate the proportion $M/T = \sqrt{\frac{3}{2}}/\pi = 0.389$ found analytically.

7.2.3. For Machine Learning Approaches

For Strategy using Incremental Update Rule

The strategy IUR, Eq. (6.24), has only one parameter, the step size γ. Fig. 7.5 (left, bottom) shows the evolution of the investment propensity values $q(t)$ for different γ values and for returns with no noise Fig. 7.5 (left, top). Note that for smaller γ values, there is a lag of the $q(t)$ values w.r.t the positive returns. Fig. 7.5 (right) shows the average of the agent's budget using this strategy vs. different γ values, for different amplitude noise level σ_2. In order to make a fair comparison between strategies, no strategies can include information about noise or periodicity to change their parameter values. Because of this, we conclude that a proper step size value for this strategy is of $\gamma = 0.5$.

GA Configuration

For the parameter tuning of the GA, first note that in the case of a periodically changing environment, Eq. (6.2), the optimal performance for the GA is obtained if the length of the chromosomes G is chosen to be equal to the periodicity T. Then, each gene matches with one of the returns values on the seasonal cycle, which allows the GA to converge to an optimal set of strategies more quickly.

The parameter values for which a GA finds a solution in less time most often depends on the optimization problem. Usually, when trying to configure meta-heuristic algorithms like the GA, one chooses a number of initial solutions which do not require too much computer time and memory consumption. Moreover, one usually chooses a large probability of recombination, a smaller probability for random mutations and a moderate elitism percentage [Goldberg, 1989; Michalewiçz, 1999]. For example, Fig. 7.6 plots the values of the RoI $r(t)$

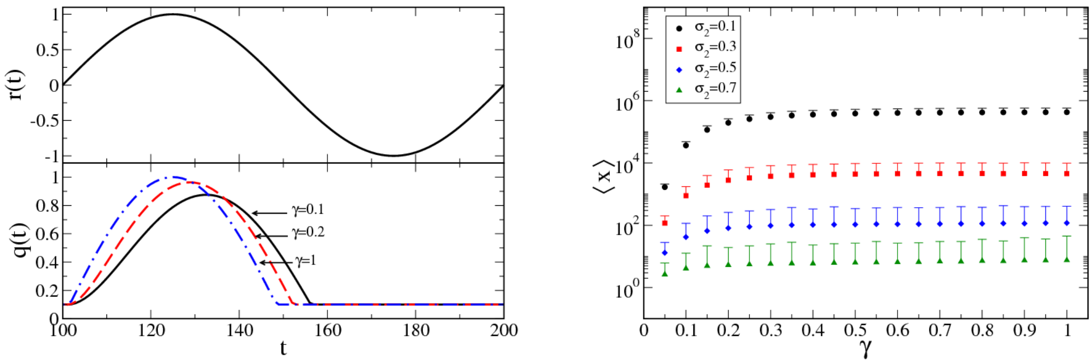


Figure 7.5.: For strategy IUR, Eq. (6.24): (left-top) RoI of the form Eq. 6.2 with $T = 100$ and (left-bottom) proportion of investment values $q(t)$; (right) average of the agent's budget for different step size γ and different amplitude noise levels. Parameters: as in Fig. 7.2.

and the corresponding proportion of investments $q(t)$ as chosen by the GA over time for different noises and for different time steps t_n. The parameter values chosen for the GA for these simulations were: number of chromosomes $C = 100$, probability of crossover $p_c = 0.8$, probability of mutation $p_m = 0.1$ and percentage of elitism $s = 0.3$. From the graph, it is visible that the behavior of the GA resembles the Ramp-Rectangle strategy presented in Section 6.4.2 and for the reader with a background in signal processing techniques, these strategies may resemble to those figures obtained when using matched filters for signal recovery (see [Turing, 1960] for further details).

It is well known that the configuration of most meta-heuristic algorithms requires both complex experimental designs and high computational effort. For this, the program *+CARPS (Multi-agent System for Configuring Algorithms in Real Problem Solving)* [Monett, 2004a,b; Monett-Diaz, 2004] was used to find the best parameters for the GA. This application uses autonomous, distributed, cooperative agents that search for solutions to a configuration problem, thereby fine-tuning the meta-heuristic's parameters.

This approach was used to configure the *GA* for periodic returns with $T = 100$, $\sigma_2 = 0$ and different levels of noise on the amplitude: $\sigma_2 = \{0.1, 0.3, 0.5, 0.7\}$. In this process, four GA parameters were optimized: the population size C, the crossover probability p_c, the mutation probability p_m and the elitism size s. Their intervals of definition, in which the most acceptable GA configurations should be found, were set as follows: $C \in \{50, 100, 200, 500, 1000\}$, $p_c \in [0.0, 1.0]$, $p_m \in [0.0, 1.0]$, and $s \in [0.0, 0.5]$, respectively.

The agents in +CARPS apply a Random Restart Hill-Climbing approach and they exchange their current best solutions to the problem in the process. Furthermore, the evaluation of the GA with a particular configuration is repeated five times in order to cope with its stochastic nature. Table 7.1 presents the best configurations obtained for the GA according to their fitness using the periodic returns previously mentioned. Notice that the fitness of a configuration, $F(\mathrm{Cfg_i})$ for $\mathrm{i} = \{1, 2, 3\}$, corresponds to the inverse value of the

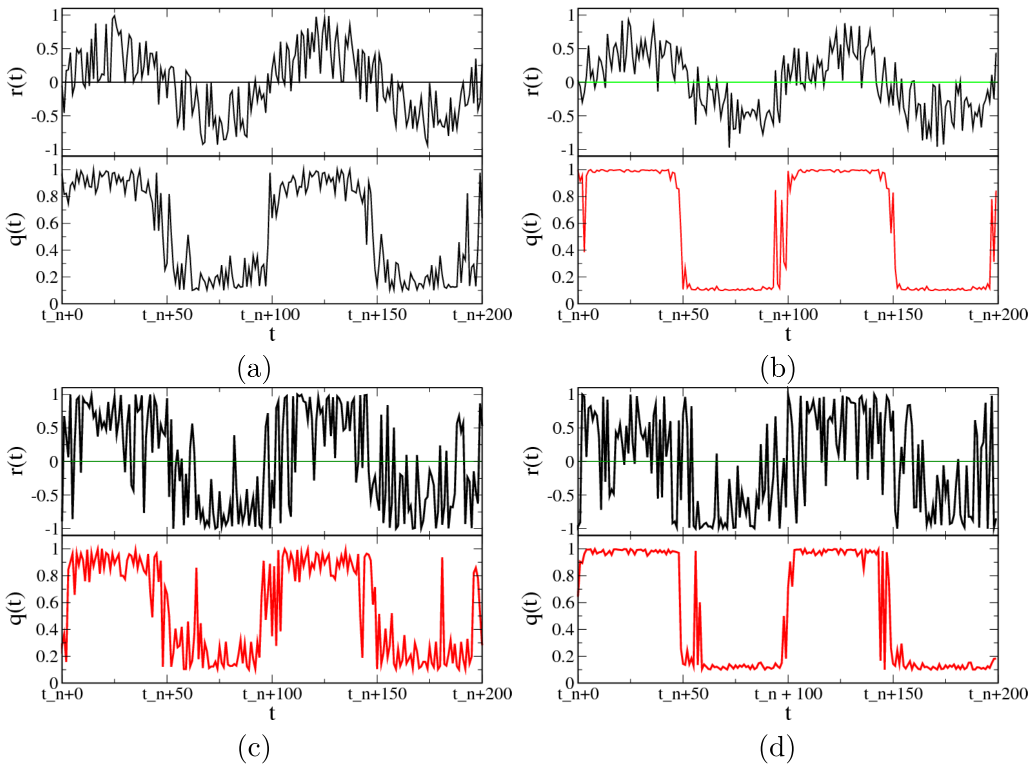


Figure 7.6.: Values of the return $r(t)$ and the proportion of investment $q(t)$ as chosen by the GA for RoI with different types of noise and for different times during the simulation: (a) amplitude noise, $\sigma_1 = 0, \sigma_2 = 0.5$, $t_n \approx 10,000$, (b) amplitude noise, $\sigma_1 = 0, \sigma_2 = 0.5$, $t_n \approx 100,000$, (c) phase noise, $\sigma_1 = 0.5, \sigma_2 = 0$, $t_n \approx 10,000$, (d) phase noise, $\sigma_1 = 0.5, \sigma_2 = 0$, $t_n \approx 100,000$.

sum of fitness values of the chromosomes with the highest fitness in each generation $f_j(\tau)$:

$$F(\mathrm{Cfg_i}) = \left[\sum_{m=0}^{\tau} \max_{j=1,\ldots,C} f_j(m)\right]^{-1} \tag{7.10}$$

Configuration Cfg_3 was used to configure the GA during the simulations performed for the periodic returns. It can be observed in Table 7.1 that the following is desired for better performance of the evolution-based strategy : large individual population, high crossover probability, low mutation probability and moderate elitism size.

7.2.4. For Daring Behavior

The same approach was used to find the optimal parameters values for strategies using a daring behavior (*RS*) Eq. (6.23) for different amplitude noise levels. For example Fig. 7.7 shows the most proper memory size values for strategies *MA* and *MLS*. Note that for the *MA* strategy as well as for *RA* behavior, the memory size remains small. However, for the *MLS* strategy and recalling our simulation results and derivation for *RA*, we note that the best memory size for a *RS* behavior differs from the previous, which in this case is $M \approx 25$. Based on visual impression, both *MA* and *MLS* strategies, like the *RA* behavior, when used with a *RS* behavior, lead to fewer fluctuations in the budget. Furthermore, some

Table 7.1.: Best obtained GA configurations when using +CARPS for no phase noise, $\sigma_1 = 0$, and different amplitude noise level σ_2.

Noise level	$\sigma_2 = 0.1$	$\sigma_2 = 0.3$	$\sigma_2 = 0.5$
Number of chromosomes	$C = 1000$	$C = 500$	$C = 1000$
Probability of crossover	$p_c = 0.830$	$p_c = 0.951$	$p_c = 0.70$
Probability of mutation	$p_m = 0.0324$	$p_m = 0.062$	$p_m = 0.010$
Percentage of elitism	$s = 0.0333$	$s = 0.432$	$s = 0.030$
Fitness	$F(\text{Cfg}_1) = 0.05276$	$F(\text{Cfg}_2) = 0.05436$	$F(\text{Cfg}_3) = 0.05340$

other experiments were performed for the *MLS* strategy with a *RS* behavior for different periodicities T and different noise levels. From these experiments it can be seen that the best memory size is:

$$M^\star \approx T/4. \tag{7.11}$$

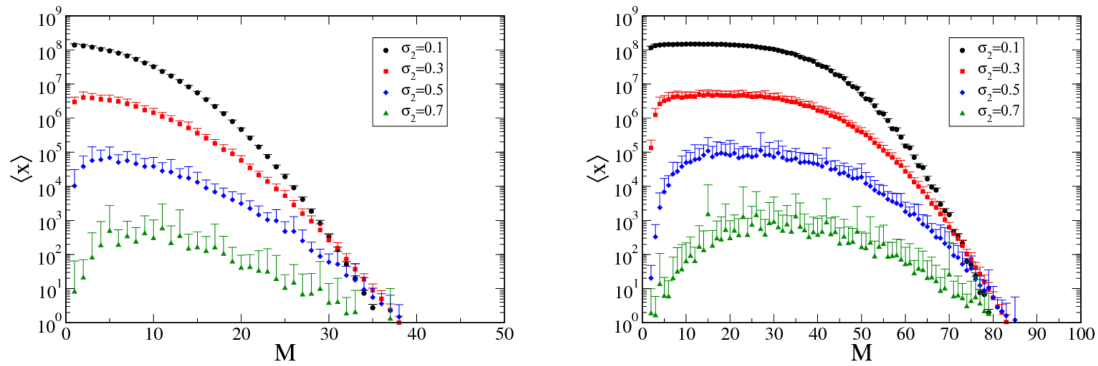


Figure 7.7.: Average of budget for strategies: (left) *MA* and (right) *MLS* for different memory sizes and different amplitude noise levels. Parameters: as in Fig. 7.2.

A number of simulations for the strategy *IUR* were also performed to find the best parameter values when a RS behavior is assumed. The results obtained lead to similar optimal parameter values as for *RA* behavior, as it can be seen in Fig. 7.8 for different amplitude noise levels. From an engineers' approach, the optimal γ value for the *IUR* strategy with *RS* behavior is considered to be the same as for the *RA* behavior, i.e. $\gamma = 0.5$.

7.3. Comparison of Results

In this section, the performance of all investment strategies presented in the previous chapter for RoI with periodicity are compared for different types and levels of noise. For the comparison, a set of agents is considered each one using one of the following strategies: Constant Proportion Of Investment *CP* Eq. (6.7), Square-Wave *SW* Eq. (6.13) (a particular case of the strategy Ramp-Rectangle *RR* Eq. (6.8) with $\triangle h = 1$), Moving Average *MA* Eq. (6.14), Moving Least Squares *MLS* Eq. (6.17), Iterative Update Rule *IUR* Eq. (6.22)

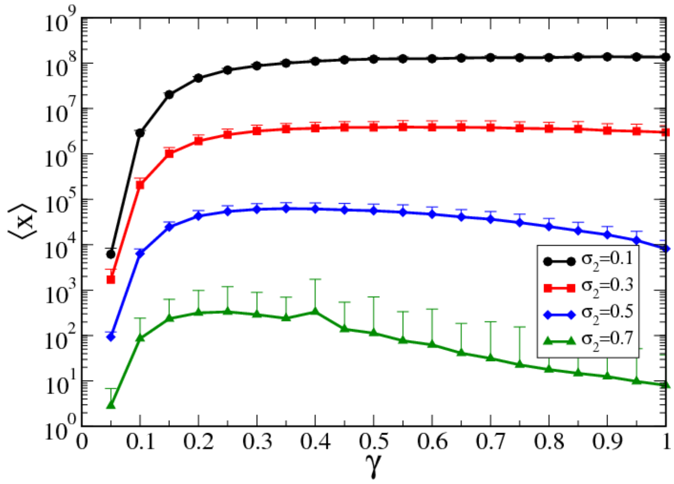


Figure 7.8.: Average of agent's budget for strategy IUR, Eq. (6.23), for different step size γ and for different amplitude noise levels. Parameters: as in Fig. 7.2.

and Genetic Algorithm *GA* Eq. (6.26).

7.3.1. Assumptions and Simulation Parameters

For the comparison, the following two important assumptions are considered: first, all agents receive the same RoI at a particular time, i.e. the fact that some agents win or lose more than others is influenced only by their different strategies for determining the correct proportion of investment value; second, all agents use the optimal parameter values of their respective strategies.

A number of $N = 100$ trials of the same experiment are performed. For each trial, a RoI with the same parameter values is considered. At the end of each cycle of the RoI, the average of the budget in the 100 trials is calculated. This is done for a large number of time steps, i.e. $t = 10^5$. Two different experiments are considered. In the first experiment, the level of the amplitude noise σ_2 is varied and the noise on the phase level is fixed to $\sigma_1 = 0$. In the second experiment, the level of the phase noise σ_1 is varied and the amplitude noise is fixed to $\sigma_2 = 0$. Finally, these two experiments are performed for both daring and cautious behaviors for the prediction of the RoI. This gives us four variants of the simulations.

7.3.2. Simulation Results and Interpretation

Fig. 7.9 shows the result of these simulations by plotting the average budget resulting from the different strategies against the noise level for each of the four variants of the simulations.

The following results can be seen from these graphs [Navarro-Barrientos et al., 2008a]:

- For all variants of the simulations, the CP strategy is the worst strategy and the SW strategy is the best strategy. This is not surprising – the CP strategy always puts a constant proportion of the budget at stake which yields a win when the return is

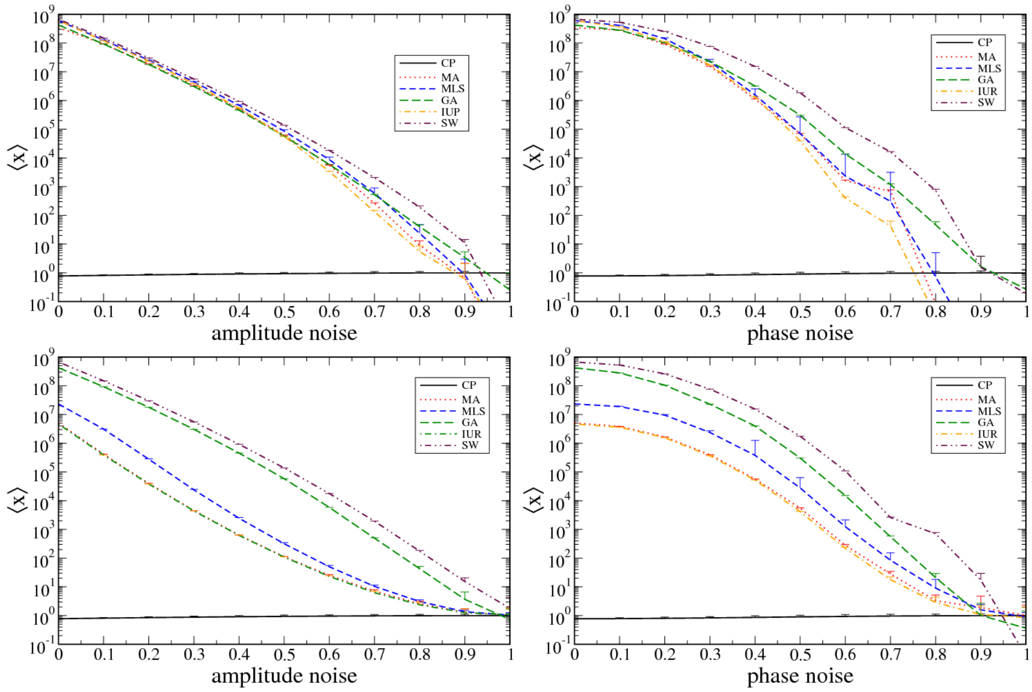


Figure 7.9.: Average budget over $N = 100$ trials, for agents using strategies CP, MA, MLS, GA, IUR and SW, over $t = 10^5$ time steps. Agents use optimal parameter values for RoI with periodicity $T = 100$ and different noise levels: (top-left) different amplitude noise levels $\sigma_2 \in (0, 1)$ and no noise on the phase $\sigma_1 = 0$ with an *RS* (daring) behavior; (top-right) different phase noise levels $\sigma_1 \in (0, 1)$ and no amplitude noise $\sigma_2 = 0$ with an *RS* (daring) behavior; (bottom-left) different noise on amplitude levels $\sigma_2 \in (0, 1)$, and no phase noise $\sigma_1 = 0$ with an *RA* (cautious) behavior and (bottom-right) different phase noise levels $\sigma_1 \in (0, 1)$ and no amplitude noise $\sigma_2 = 0$ with an *RA* (cautious) behavior.

positive, but a loss when the return is negative even though $\langle 1 + r(t)q(t) \rangle = 1$. This leads to a loss in budget over time, as is well known for multiplicative stochastic processes, see Section 3.2. Moreover, the SW strategy has complete knowledge about the periodicity of the return, so it can always invest the appropriate amount.

- For all strategies, the average budget decreases with increasing noise. This is expected: with increasing noise, the accuracy of the predictions made by the agents decreases, and thus they cannot necessarily chose the appropriate proportion of investment in the action.

- There are no significant cross-overs of the performance of different strategies. In general, this implies that if a strategy s_1 performs better than a strategy s_2 for a given noise level σ_a (either on the phase or on the amplitude), s_1 can be expected to perform better than s_2 for a different noise level σ_b. Consequently, the choice of strategy is independent of the noise in the return – a good strategy is a good strategy for all noise levels and a bad strategy is a bad strategy for all noise levels. However, for low noise levels, the GA is slightly outperformed by the other strategies – this is due to the intrinsic stochastic nature of the algorithm. For the same reason, this algorithm performs better for high noise levels.

- From the range of strategies employed, the simple strategies (MA, MLS, IUR) were almost always outperformed by the complex one (GA). However, some researchers have shown that this needs not necessarily be the case for more complex scenarios like in artificial exchange markets [Farmer, 2001; Farmer et al., 2005].

- The GA approaches the optimal strategy. Considering that the GA does not have an a priori behavior, it is interesting to realize that it discovers the optimal strategy – investing the maximum when, at a particular time t in the period, the probability of winning is higher than losing and vice versa – on its own.

- For low levels of noise, the daring behavior clearly outperforms the cautious behavior: always investing the maximum when a positive return is expected and investing the minimum when a negative return is expected outperforms investing a quantity proportional to the expected return. This may seem counter-intuitive. Even though the best strategy is to still invest the maximum when there is a slightly larger probability that $r(t) > 0$ than that $r(t) \leq 0$. All strategies except for the GA fail to predict the exact probabilities of $r(t) > 0$ and $r(t) \leq 0$ with enough accuracy to determine how to properly invest. Since the GA does not exhibit the prediction-action behavior, it is better than the other strategies.

- For phase noise, the average budget obtained is roughly comparable than for amplitude noise, although the differences between strategies are greater for phase noise than for noise on the amplitude.

7.4. Summary

In this chapter, we have compared the performance of different investment strategies that can be applied by agents in an investment market scenario with periodic returns and different types and levels of noise. For each strategy, the parameter values that lead to larger average budget values were found by means of simulations. The performance of these strategies – the respective average budget growth over a certain number of time steps – was compared and the results were analyzed.

Probably, the most interesting fact in this study is that, while it seems intuitive to invest an amount proportional to the expected return, this is not the approach which yields the greatest budget growth over time. On the contrary, a “gambling” approach of always investing the whole budget as soon as the probability of a positive return is greater than the probability of a negative, results in a greater budget growth over time. Surprisingly, the genetic algorithm discovers this strategy and for these reasons, it is also the algorithm which best approximates the optimal strategy.

Finally, we note also that beside *+CARPS*, different methods could be used for the meta-heuristics used for the Genetic Algorithm, for example the use of ant colonies, see [Dorigo and Caro, 1999].

8. Adaptive Investment Strategies for Changing Periodic Environments

This chapter presents an adaptive investment strategy based on a genetic algorithm for cyclic changing environments and compares its performance in an environment characterized by returns with changing periodicities.

8.1. Introduction

The main goal of the previous chapter was to show how the complexity of the agent, the memory size and the available information influence the performance of the agent for different types and levels of noise in stationary periodic environments. The performance of the different types of strategies was based on the total budget obtained by the agent after a number of time steps. In the last chapter, it was assumed that all strategies knew the periodicity of the returns. In this chapter, these previous investigations are extended to find the best investment strategy for returns with changing periodicity, i.e. non-stationary periodic environments.

It was shown in Section 7.3 that the *GA* strategy presented in Section 6.6.2 outperformed the other strategies. In this chapter, the complexity of the strategy *GA* is extended by allowing the chromosomes to have a different length and considering more complex cross-over and mutation operators so that the algorithm may find the best investment strategies for the current periodicity and noise level of the returns. In other words, in this chapter, an adaptive investment strategy based on a *Genetic Algorithm* for changing periodic environments is presented, which for simplicity is called *GACE* (*Genetic Algorithm for Changing Environments*).

For the testbed scenario, the investment model described in Eq. (6.1) is used to model an agent which decides at every time step the proportion of wealth to invest in a risky asset, keeping the rest of the budget in a risk-free asset. Every investment is evaluated in the market via a stylized return on investment function (RoI), which is modeled by a stochastic process with unknown periodicities and different levels of noise. Like before, two reference strategies are considered; these represent the case of agents with zero-knowledge and complete-knowledge of the dynamics of the returns (see Section 6.4). From the strategies based on technical analysis methods presented in Section 6.5, it was shown in Section 7.3 that the investment strategy based on *Moving Least Squares* (MLS) performed better than the other technical strategies. Because of this, in this chapter, only the MLS strategy is considered.

The structure of this chapter is as follows: Section 8.2 presents some background on adaptive investment strategies for changing environments and Section 8.3 presents the adaptive investment strategy called *GACE* that extends the strategy based on a Genetic Algorithm presented in Section 6.6.2. In Section 8.4 we present the results obtained for different computer experiments. In order to understand the way in which GACE adapts to its environment, the computer experiments are performed in a controlled scenario where the dynamics of the environment are known. For that reason, it is assumed that the risky asset is modeled by a stochastic process with changing periodicity and different levels of noise, i.e. stylized exogenous returns, presented in Section 8.4.1. The experiments to account for the performance of the investment strategies are divided into two sections. First, in Section 8.4.2, the performance of GACE for returns with fixed periodicity is revisited and investigated in more detail and in Section 8.4.3, the performance of the different strategies is compared for returns with changing periodicity.

8.2. Background

Many researchers have used different machine learning methods to find good investment strategies in different type of stochastic environments. For example, Magdon-Ismail et al. [2001] used neural networks to find patterns from financial time series, where the main goal is to find changes in volatility. Moreover, Geibel and Wysotzki [2005] proposed the use of a risk-sensitive reinforcement learning algorithm to find the best policy for controlling under constraints. The authors applied this approach to control a feed tank with stochastic inflows. Other techniques from machine learning that are frequently used for investment decision problems are those based on *evolutionary computation*. For example, those using *genetic programming* and *genetic algorithms* for portfolio management, inducing rules for bankruptcy prediction and assigning credit scoring [Dawid, 1999]. Some investment strategies based on genetic programming techniques usually lead to profitable trading strategies, however, they usually find strategies which are difficult to understand and sometimes they cannot be funded [Jiang and Szeto, 2003; Neely et al., 1997; Schulenburg and Ross, 1999, 2001]. Even though investment strategies that are based on genetic algorithms may be also difficult to abstract and to explain, we believe that they are more natural and understandable than those using genetic programming techniques [Drake and Marks, 2002]. However, many of these approaches are applied to environments that are stationary; this means that some of them cannot be directly applied to changing environments. There are some researchers who have investigated the use of genetic algorithms in changing environments [Branke, 1999; Grefenstette, 1992]. However, to our knowledge, standard genetic algorithms have not been applied specifically to the problem of controlling the proportion of investment in investment environments with changing periodicities.

The investigations in this chapter may also draw interest in the research area of pattern recognition of time series. In particular, for the cases in which there is no prior knowledge of the existence of a periodic signal or of its characteristics, see [Alvarez et al., 2001; Szpiro, 1997]. Note that with some changes on the proposed adaptive algorithm *GACE*, a useful algorithm could be proposed for the detection and measurement of periodic signal in time series.

8.3. Adaptive Investment Strategy

In this section, we present an adaptive investment strategy based on a *Genetic Algorithm* for controlling proportions of investment in changing periodic environments. For simplicity, we call this strategy *Genetic Algorithm for Changing Environments* (GACE).

In Section 6.6.2, a strategy based on a Genetic Algorithm (GA) was presented. This strategy was able to find the best proportion of investment for returns with fixed periodicity.

In this section, this approach is extended for returns with changing periodicity [Navarro-Barrientos, 2007, 2008a,b].

8.3.1. Encoding Scheme

As, in Section 6.6.2, the GA is populated with chromosomes $j = 1, ..., C$, where each chromosome j has an array of genes g_{jk} where $k = 0, ..., G_j - 1$, and G_j is the length of the chromosome j. Note that now, the chromosome has a variable length in comparison with the specification of the chromosome with fixed length in Section 6.6.2. Thus, the length of a chromosome is now assumed to be in the range $G_j \in (1, G_{\max})$, where $G_{\max}$ is a parameter that specifies the maximal allowed number of genes in a chromosome. As it was noted previously, the values of the genes could be binary, but for programming reasons we use real values [Michalewiçz, 1999]. Again, each chromosome j represents a *set of possible strategies* of an agent, i.e. each g_{jk} corresponds to an proportion of investment.

8.3.2. Fitness Evaluation

Each chromosome j is evaluated after a given number of time steps by a *fitness function*, $f_j(\tau)$, which is defined as follows:

$$f_j(\tau) = \sum_{k=0}^{G_j-1} r(t)\, g_{jk}\ ; \quad k \equiv t \bmod G_j, \tag{8.1}$$

where τ is a further time scale in terms of generations. When a generation is completed, the chromosomes' population is replaced by a new population of better fitted chromosomes with the same population size C.

As noted previously, every g_{jk} is multiplied by a different value of $r(t)$ over time. This is done in accordance with the investment model, Eq. (6.1).

8.3.3. Selection of a New Population

If we assume that the chromosomes have a fixed length $G_j = G_{\max}$, as in Section 6.6.2, then the most proper number of time steps, t_{eval}, that have to elapse in order to evaluate all chromosomes' genes is $t_{\text{eval}} = G_{\max}$. In other words, the number of time steps needed to evaluate the population is equal to the fixed length of the chromosomes.

Moreover, it can be shown that the population converges faster towards optimal proportions of investment if the length of the chromosomes is equal to the periodicity of the returns, $G_{\max} = T$. However, this previous assumption corresponds to the ideal case where the agent knows a priori the periodicity of the returns and sets the length of all chromosomes to the value of the periodicity. Hence, the agent selects a new population after all genes of all chromosomes are evaluated. Now, if the chromosomes have different lengths the following question may arise: *After how many time steps t_{eval} should a new generation of chromosomes be obtained?* In the following, we propose different approaches to answer this question.

Time steps for evaluation

Different approaches can be proposed to determine the number of time steps t_{eval} that should elapse to select a new generation of chromosomes. As mentioned above, the simplest approach, called *GMaximum*, is to select a new population after a fixed number of time steps $t_{\text{eval}} = G_{\max}$ have elapsed, where $G_{\max}$ is the maximal possible length of a chromosome. Note that in this approach, all genes of all chromosomes in the population are being evaluated. However, some computer experiments showed that such an approach leads to slow convergence of the population. A different approach is to choose the number of time steps for evaluation according to the length of the best chromosome in the population. This approach is called *GBestSelected*, and it can be expressed mathematically as follows:

$$t_{\text{eval}} = G_l \ ; \quad l = \arg\max_{j=1..,C} f_j(t'_{\text{eval}}), \tag{8.2}$$

where t'_{eval} is the number of time steps that the population has been evaluated in the current generation. Some computer experiments showed that this approach leads to a faster convergence of the population than when using GMaximum; however, if the length of the best chromosome in the previous generation happens to be very large, this would lead to a larger number of time steps in which the agent would be using this strategy only. This would be disadvantageous for the agent if the environment has changed and the current strategy leads to loses instead of profits for the current returns. Computer simulations showed that a better approach is to choose the number of time steps needed for evaluation according to the length of the best chromosome at every time step t. This approach is called *GBestCurrent*, and can be expressed mathematically as follows:

$$t_{\text{eval}} = G_l \ ; \quad l = \arg\max_{j=1,..,C} f_j(t). \tag{8.3}$$

Note that the last two approaches GBestSelected and GBestCurrent have the disadvantage that they do not assure that all genes of all chromosomes are being evaluated; however, from our point of view, good chromosomes would lead to a higher level of fitness than bad ones from the very beginning of the evaluation. By coincidence, it can happen that poorly fitted chromosomes lead to large profits in the beginning; however, in the long run, only the best chromosomes will survive. Unless otherwise indicated, we assume in the following that the approach GBestCurrent is used for the evaluation of the population.

Elitist and Tournament Selection

Once the time has come to select a new population, the question is: *how is a new population determined?* As we did before for the strategy GA, after calculating the fitness of each chromosome according to Eq. (8.1), the best chromosomes are selected by applying elitist and tournament selection of size two (see Section 6.6.2).

Crossover and Mutation Operators

Once two chromosomes have been selected by means of the tournament selection, a simple crossover operator exchanges genetic information between the two chromosomes, whatever their sizes, by finding the cross-point with respect to the size of the shortest chromosome. This is done by randomly selecting the cross-point or cut-point c_p from the shortest chromosome and exchanging with probability $p = 0.5$, the genetic material above or beyond this cross point in the shortest chromosome with its counterpart in the largest chromosome. However, this would mean that those genes in the largest chromosome beyond the length of the shortest chromosome would be disregarded.

The limitations of conventional crossover in GA with variable length has already been addressed by some authors [Harvey, 1992], where neural networks or hierarchical tree-structures are used to determine which genes should be exchanged between the chromosomes. However, for the purpose of this chapter and for the sake of simplicity, we propose a modification of the standard GA crossover operator that better suits our demands.

Thus, we propose the use of a crossover operator called *Proportional Exchange Crossover* (PEC) operator, which basically shrinks or stretches the genetic information between the pair of chromosomes proportionally to their length. Basically, the crossover operator PEC randomly selects the range of genetic information to be exchanged between two chromosomes and shrinks/stretches the genetic information from the longest/shortest to the shortest/longest chromosome, respectively.

Algorithm 3 shows the PEC algorithm for all pair of parent-chromosomes being selected via tournament selection. Note that a chromosome j is saved in an array with indexes in the range 0 to $G_j - 1$.

Algorithm 3: Proportional Exchange Crossover (PEC) operator

1 **foreach** *pair of parent-chromosomes* **do**
2   determine the shorter and the larger parent-chromosomes pa_s and pa_l with sizes G_s and G_l respectively
3   select randomly the cross-point $cp_s \in \mathbb{Z}$ for the short parent-chromosome: $cp_s \sim U(0, G_s - 1)$
4   find the cross-point $cp_l \in \mathbb{Z}$ for the large parent-chromosome: $cp_l = \frac{G_l\, cp_s}{G_s}$
5   determine the proportion $R \in \mathbb{Z}$ between the two sizes: $R = \frac{cp_l}{cp_s}$
6   create two arrays, ch_s and ch_l, for the short and large children-chromosomes
7   with equal probability choose the side for the crossover operation
8   **if** *crossover on the left side* **then**
9   stretch the genetic material from pa_s to ch_l as follows:
10   **for** $m = 0$ *to* $cp_s - 1$ **do**
11   **for** $n = 0$ *to* $R - 1$ **do**
12                 $ch_l[m \cdot R + n] \leftarrow \mathrm{stretch}(pa_s, m, R)$
13   **end**
14   **end**
15   shrink the genetic material from pa_l to ch_s as follows:
16   **foreach** $m = 0$ *to* $cp_s - 1$ **do**
17             $ch_s[m] \leftarrow \mathrm{shrink}(pa_l, m, R)$
18   **end**
19   **else**
20   stretch as in line 12 but for the range $m = cp_s$ to $G_s - 1$.
21   shrink as in line 17 but for the range $m = cp_s$ to $G_s - 1$.
22   **end**
23   copy directly the rest genetic material from the parents to the children chromosomes.
24 **end**

Note that different functions could be considered for the transformation of the genetic material between chromosomes with different length. For simplicity, in our computer experiments we consider a PEC operator based on averaging and copying the genetic material of the parent-chromosomes. This means that in Algorithm 3, for the implementation of

GACE: in line 12 the following function is considered:

$$\text{stretch}(pa, m, R) = pa[m]. \tag{8.4}$$

This function simply copies the genes from the short parent-chromosome to the large child-chromosome; furthermore, in line 17 the following function is considered:

$$\text{shrink}(pa, m, R) = \frac{1}{R}\sum_{i=m}^{m+R} pa[i]. \tag{8.5}$$

This function performs an average of the genetic material. A more interesting option for these transformations could be based on the dynamic time warping algorithm [Sankoff and Kruskall, 1983], which is usually used for calculating the similarity between two signals. With some modifications, this algorithm could be used to stretch or shrink the genetic material proportionally to the original material; however, this is far from the scope of this chapter.

To illustrate how the PEC operator works, Fig. 8.1 shows a graphical representation of PEC applied to the left side of the cross-point. In this example, the cross-point of the shortest chromosome is $cp_s = 3$. Consequently, using line 4 in Algorithm 3, we find that the cross-point for the largest chromosome corresponds to $cp_l = 6$. In this example, the genes to the left of the shorter "parent" chromosome are generalized into the larger "child" chromosome and the genes to the right of the cross-point are directly copied into the shorter "child" chromosome. The same occurs for the genes in the larger "parent-chromosome" with the main difference being that the value of the genes to the left are averaged and not generalized. In the same manner, the cross-points in the "parent" chromosomes are determined if the right side of the crossover is selected.

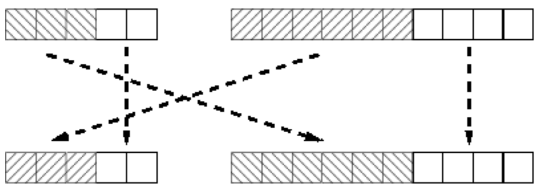


Figure 8.1.: Example of the *Proportional-sized Exchange Crossover* (PEC) operator. With probability $p = 0.5$ the left side of the cut point is selected for exchange.

Now, to make sure that diverse chromosome lengths are present in the population of chromosomes, a mutation operator is introduced for the length of the chromosome G_j. For this, a new length is drawn randomly and the genetic information of the chromosome is proportionally scaled to the new length. In other words, this operator mutates the length of the chromosome G_j with probability p_l leading to a new enlarged or stretched chromosome. The algorithm used for the mutation of the length of the chromosome is based on the same principle as the PEC operator, shrinking or stretching the genetic material.

Thus, the combination of the PEC operator and the mutation in the chromosomes' length may help to determine the optimal proportions of investment and the periodicity (or patterns) of the returns, respectively. After the crossover and length-mutation operators are applied, the typical gene-mutation operator is applied. This means that with a given mutation probability $p_m \in U(0, 1)$, a gene is to be mutated by replacing its value by a random number from a uniform distribution $U(q_{\min}, q_{\max})$.

In summary, given a population with C chromosomes, to obtain a new generation of chromosomes, one needs to do the following:

1. Apply the elitist operator to select the best s percent of the population which are directly included in the new population.

2. Apply the tournament selection operator to the current population to select two "well-fitted" parents.

3. With probability p_c, apply the PEC crossover operator to the two selected parent-chromosomes leading to two new children-chromosomes.

4. With probability p_l, apply the length-mutation operator to the two children chromosome in order to ensure length diversity in the new population.

5. With probability p_m, apply the gene-mutation operator to the two children which are then included in the new population.

6. Finally, repeat steps (2) to (5) until the new population has the same number of chromosomes as the original population.

Strategy Selection and Initialization

Once a new population has been obtained, we need to answer the following question: *how does the agent update its actual proportion of investment $q(t)$?*

For every new generation, the agent takes the set of strategies g_{jk} from the chromosome j with the largest fitness in the previous generation.

$$q_i(t) = g_{lk} \quad \text{with } l = \arg\max_{j=1,\ldots,C} f_j;\ k \equiv t \bmod G_l \tag{8.6}$$

For the initialization, each g_{jk} is assigned a random value drawn from a uniform distribution: $g_{jk} \sim U(q_{\min}, q_{\max})$. The length of the chromosomes can be set initially to a fixed number of genes or it can be determined randomly. For the latter, each G_j is initialized with an integer random value drawn from a uniform distribution $U(1, G_{\max})$, where $G_{\max}$ is the maximum permitted chromosome length.

8.4. Experimental Results

In this section, we systematically analyze the performance of the strategies presented previously. For this, in Section 8.4.1; the environment for the agent is presented. Afterwards, we investigate the parameter tuning of the Genetic Algorithm. For this, the complexity of the operators and the environment is increased systematically. Finally, we compare the performance of the adaptive strategy against some of the strategies presented in the previous chapter.

8.4.1. Artificial Returns

Recalling the artificially generated returns presented in Eq. 6.2, in this chapter we consider returns with a non fixed periodicity $T(t)$ and for simplicity, only amplitude fluctuations are assumed [Navarro-Barrientos, 2008b]:

$$r(t) = A \sin\left(\frac{2\pi}{T(t)}\, t\right) + \sigma_2 \xi_2, \tag{8.7}$$

where the A is the amplitude of the periodic returns with $A = (1 - \sigma_2)$, $\sigma_2 \in (0, 1)$ is the amplitude noise level and ξ_2 corresponds to a random number drawn from a uniform distribution $\xi_2 \in U(-1, 1)$. The periodicity of the returns has the following dynamic:

$$\text{if } t < t' \Rightarrow T(t) = T(t-1) \tag{8.8}$$

$$\text{else} \Rightarrow \begin{cases} T(t) = \tilde{T} \\ t' = t + \tilde{t}, \end{cases} \tag{8.9}$$

where t' determines the number of time steps that the current periodicity will be present in the returns (initially $t' = 0$); both $\tilde{T}$ and $\tilde{t}$ are random numbers drawn from the uniform distributions $U(0, T_{\max})$ and $U(0, t_{\max})$, respectively.

Thus, σ_2 accounts for the fluctuations in the market dynamics on the amplitude of the RoI, $T_{\max}$ accounts for the largest possible periodicity and $t_{\max}$ accounts for the maximal number of time steps a periodicity can elapse. Note that with respect to the artificial returns considered in Section 6.3, for simplicity, only fluctuations on the amplitude are considered, i.e. fluctuations on the phase are not considered. Fig. 8.2 shows an example of the RoI for different noise level σ_2.

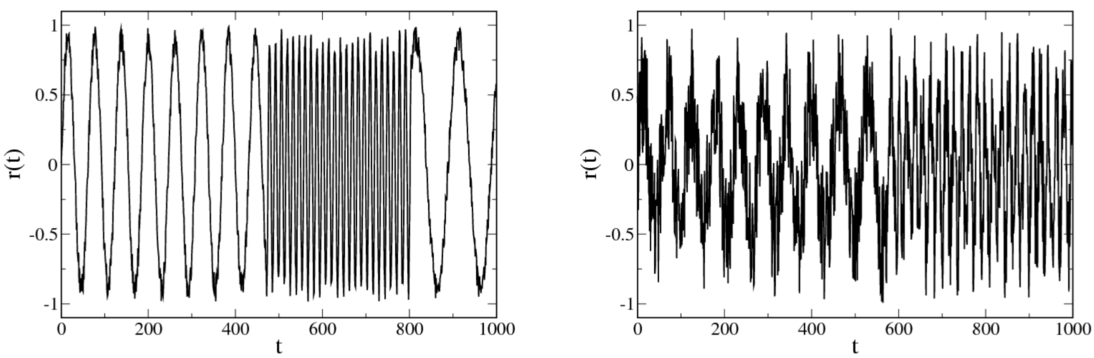


Figure 8.2.: Periodic RoI, $r(t)$, Eq. (8.7) for different amplitude fluctuations: (left) $\sigma_2 = 0.1$, and (right) $\sigma_2 = 0.5$. Further parameters: $T_{\max} = 100$ and $t_{\max} = 1000$.

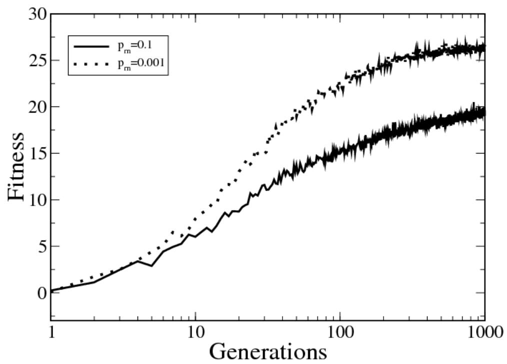
8.4.2. Convergence for RoI with fixed periodicity

To investigate the convergence of the adaptive strategy proposed in Section 8.3, we start with a simple scenario in which returns have a fixed periodicity. In Section 8.4.3 we consider a more challenging scenario in which returns have a changing periodicity.

Evolution of the Fitness

For the sake of completeness, we investigate the performance of the adaptive strategy for fixed periodicity $T(t) = T$ in Eq. (8.7) in more detail. We start our analysis of the convergence of GACE by studying the evolution of the fitness of the whole population of strategies and the fitness of the best strategy. For these computer experiments, the parameter values in Table 7.1 are considered together with a zero probability of length mutation $p_l = 0$ and the initial length of the chromosomes is fixed at $G_j = G_{\max} = T$.

Fig. 8.3 (left) shows the evolution of the average fitness of the chromosomes in the population for different mutation rates over the course of several generations. Moreover, Fig. 8.3 (right) shows the evolution of the average fitness of all chromosomes and the fitness of the best fitted chromosome over the course of several generations for $C = 1000$ chromosomes. Observe that the rate $p_m = 0.01$ used for Fig. 8.3 (right) leads to larger average fitness in the population than a higher or lower mutation rate as showed in Fig. 8.3 (left); however, note that the fitness of the best chromosome when using $p_m = 0.001$ is almost as good as for $p_m = 0.01$. Note that in Fig. 8.3 (right), for the first 100 generations the best chromosome performs much better than all chromosomes on average; however, after 100 generations, we can see that the performance of the population converges with the performance of the best chromosome. Now, consider again Eq. (6.25) and replace g_{jk} with $q(t)$ and G_j with T. If we consider returns for $t = 100$ time steps with periodicity $T = 100$ and no noise, it can be shown that the strategy SW would lead to a fitness of $f(\tau) = 28.63$. Note that this is not much larger than the fitness obtained with GACE.



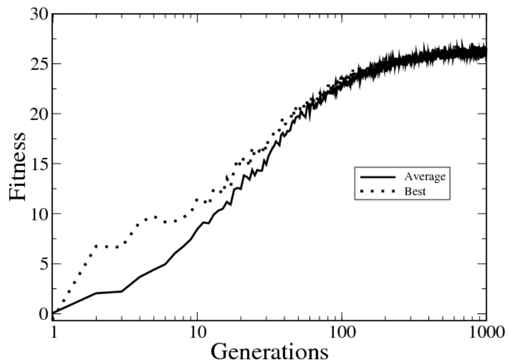


Figure 8.3.: Performance graphs for the GACE strategy over the course of multiple generations for $C = 1000$ chromosomes: (left) average fitness for different mutation probabilities p_m and (right) convergence of the population showing the average fitness for all chromosomes against the fitness of the best chromosome for $p_m = 0.01$. Further parameters: $T = 100$ and $\sigma_2 = 0.1$.

Evolution of the Budget

Another way in which to account for the convergence of the adaptive strategy is to investigate the evolution of its performance over time together with the performance of other strategies. In order to assure fair comparison between the strategies, as we did in Section 7.2, we need to find the best parameter values for the strategies. For this, we consider both reference strategies CP (Eq. (6.7)) and SW (Eq. (6.13)) with parameters $q_{\min} = 0.1$ and $q_{\max} = 1.0$. For the strategy MLS (Eq. (6.19)), we assume again that the agent has access to some information about the returns, i.e. the agent knows the periodicity T of the returns. This means that the agent needs to determine the best memory size M based on the known periodicity of the returns. For this, we assume an agent using the optimal memory size presented in the previous chapter, Eq. (7.9). For the GACE strategy the parameters in Table 7.1 are used and the initial length of the chromosomes are chosen at random from a uniform distribution. However, this means that we need to define the range of possible length values. For our implementation of the GA to work properly, the unknown periodicity needs to be in the range of possible length values. For simplicity, in our experiments, we let the range to be larger than the periodicity of the returns. However, we note that this parameter could be determined by the GA itself if we include extra genes in the chromosome to track for a proper range. Another possibility could be to determine this parameter by means of statistical properties of the returns, like the autocorrelation function or spectral density; however, both approaches are beyond the scope of this chapter and are left for further work.

As it was done previously, a synthetic data set was generated for the returns. In these experiments, it is assumed that the agent invests in returns with periodicity $T(t) = 100$ for different noise levels. For the moment assume that the length of the chromosomes is fixed to $G_j = 100$ and that a new generation of chromosomes is obtained after a number of time steps $t_{\text{eval}} = 100$ have elapsed. For the computer experiments, we let the agent use one of the strategies to invest during a number of $t = 10^5$ time steps. In order to account for the randomness of the scenario, we perform the experiment for a number of $N = 100$ trials, gathering the average budget obtained for each strategy at every 100 time steps.

Fig. 8.4 shows in a log-log plot the average budget $\langle x \rangle$ over the course of GACE's generations τ for all strategies and for returns with different amplitude noise levels. As you can see, except for the GACE strategy, all other strategies have a constant budget on average over each generation. This occurs because the average of the budget was taken at every $t_s = 100$ time step. This corresponds to the periodicity of the returns $T(t) = 100$ and to the time steps to evaluate the population of chromosomes $t_{\text{eval}} = 100$, as it was specified in our experiment parameters.

Fig. 8.4 (left) shows that after 4, 70, and 300 generations, GACE outperforms the strategies $q = 0.1$ and MLS, respectively. We note that GACE performs almost as well as the SW strategy after 400 generations. Moreover, the budget increase of the agent using GACE can be approximated by a power law for the first 100 generations; afterwards it increases logarithmically.

Fig. 8.4 (right) shows that for large amplitude noise, it takes fewer generations for GACE to outperform the strategy $q = 0.1$, but in general, more generations are needed for GACE to outperform the other strategies. We find that the budget also increases according to a power law for the first 100 generations and afterwards it increases logarithmically. We note that it would be useful to provide a formulation to characterize the average budget over the course of several generations that is obtained using the GACE strategy, however, this is left for further work.

Note that in the beginning, the length of the chromosomes is drawn randomly, however, the following question may arise: *Do the chromosomes' lengths evolve to map the periodicity of the returns?* To answer this question, we performed some computer experiments for an agent using the GACE strategy for returns with fixed periodicity $T(t) = 100$ and different noise levels. For these experiments, we assumed the parameter values specified in Table 7.1

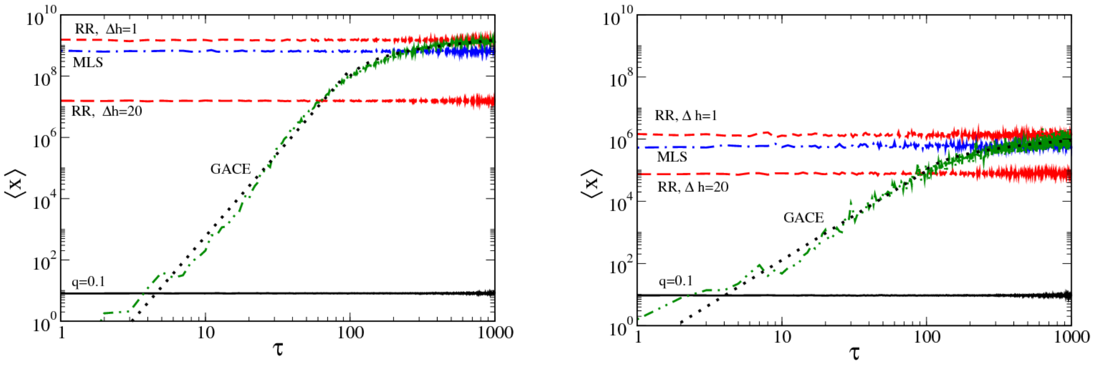


Figure 8.4.: Average budget, $\langle x \rangle$, obtained using different investment strategies over the course of generations τ of the strategy GACE, for returns with periodicity $T = 100$ and amplitude noise: (left) $\sigma_2 = 0.1$ and (right) $\sigma_2 = 0.5$.

for GACE, and we consider that the initial chromosomes' length is drawn randomly from a uniform distribution with range of values $(1, G_{\max})$, where $G_{\max} = 500$.

Fig. 8.5 shows the probability distribution of the length of the best fitted chromosomes for different noise levels and for different generations $\tau = \{5, 100\}$. It is clear that after five generations, most of the chromosomes' length have properly matched the periodicity of the returns. Interestingly, chromosomes with lengths proportional to a multiple of the periodicity are also frequent; however, the probability decreases for larger multiples of the real periodicity, which is a consequence of the better adaptation of smaller chromosomes which have found the best proportions of investment more quickly.

8.4.3. RoI with Changing Periodicity

In the previous section, we investigated the performance of the adaptive strategy in a stationary environment. In this section, we tackle a non-stationary environment. As in Section 7.2.3, in order to find the best parameters for the adaptive strategy GACE, we used the software *+CARPS (Multi-agent System for Configuring Algorithms in Real Problem Solving)* [Monett, 2004a].

The genetic algorithm was configured for periodic returns with $T_{\max} = 100$ and different levels of noise: $\sigma_2 = 0.1$ and $\sigma_2 = 0.5$. In this process, the following parameters were optimized: the population size C, the crossover probability p_c, the mutation probability p_m, the elitism size s, and the probability of length mutation p_l. The intervals of definition for the parameter were: $C \in \{50, 100, 200, 500, 1000\}$, $p_c \in [0.0, 1.0]$, $p_m \in [0.0, 1.0]$, $s \in [0.0, 0.5]$ and $p_l = \in [0.0, 1.0]$. The evaluation of the genetic algorithm was repeated five times for each configuration. Thus, the best parameter values for GACE are shown in Table 8.1.

Note that with respect to Table 7.1, for the parameter values in Table 8.1 the crossover and mutation operators are less likely to occur when recombining two parents. However, this is covered by a surprising large probability of mutation on the length of a chromosome.

Fig. 8.6 (top) shows the evolution of the average budget over time for an agent using the GACE strategy to invest in returns with changing periodicity and different noise levels. For

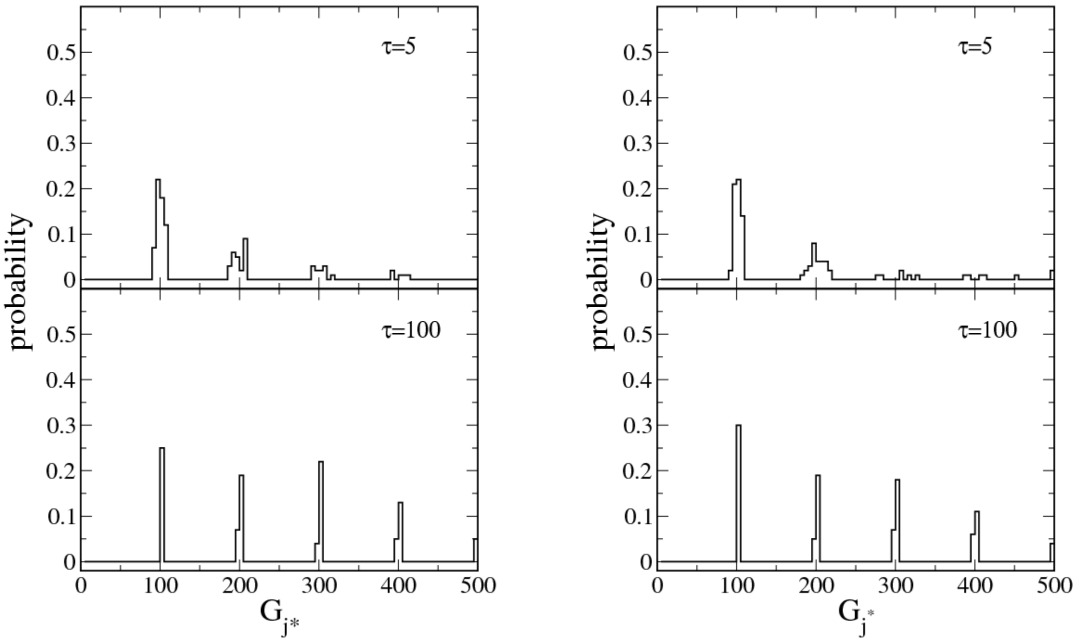


Figure 8.5.: Probability distribution of the length of the best fitted chromosomes for generations $\tau = \{5, 100\}$ for $N = 50$ trials. Returns with periodicity $T(t) = 100$ and amplitude noise: (left) $\sigma_2 = 0.1$ and (right) $\sigma_2 = 0.5$.

the sake of clarity, Fig. 8.6 (bottom) includes the corresponding periodicities of the returns for each time step.

In Section 7.2, we noted that in order to avoid overflows, we need to reinitialize the budget of the agent after every cycle of the returns. Now for RoI with changing periodicity, the budget of the agent is reinitialized to the initial budget every time the periodicity of the returns changes. This is done in order to show the increase in average budget that is achieved for a given periodicity of the RoI and not only for a cycle of the RoI as it was done in the previous chapter.

From the dynamics of the returns, Eq. (8.7), it can be seen that a change of periodicity is not performed exactly at the end of a period but at any time step. This is the reason for large increases or decreases of budget each time the periodicity of the returns changes.

Performance Comparison

In this section, we investigate the performance of the adaptive strategy with respect to the reference strategies for a non-stationary scenario. For this, we performed some computer experiments for returns with changing periodicity and different noise level. As we did in the previous sections we assumed for all strategies the parameter values $q_{\min} = 0.1$ and $q_{\max} = 1.0$. Moreover, for the MLS strategy we used Eq. (7.9) to calculate the memory size M. And for the GACE strategy we used the parameter values listed in Table 8.1 and the length of a chromosome in the range $G_j \in (1, G_{\max})$, with $G_{\max} = 200$.

In Fig. 8.7 we show the evolution of budget (top) and the corresponding periodicity of

Table 8.1.: GACE's best parameter values for RoI with changing T.

Parameter	Value
Number of chromosomes	$C = 1000$
Probability of crossover	$p_c = 0.5$
Probability of gene mutation	$p_m = 0.001$
Percentage of elitism	$s = 0.3$
Probability of length mutation	$p_l = 0.5$

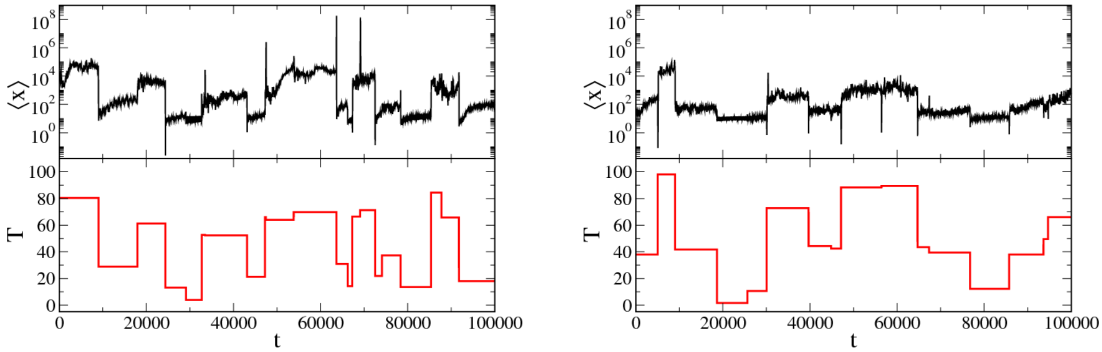


Figure 8.6.: (top) Average budget over time for $N = 50$ trials for an agent using the GACE strategy with parameter values as in Table 8.1 and the parameter values $G_{\max} = 200$. (bottom) Periodicity of the returns over time, Eq. (8.7), with parameters: $T_{\max} = 100$ and $t_{\max} = 10^4$. Both for different amplitude noise: (left) $\sigma_2 = 0.1$ and (right) $\sigma_2 = 0.5$.

the returns Eq.(8.7) (bottom), both over time for the different investment strategies and different noise levels. It is clear that the best strategy for both cases is the SW strategy, followed by the MLS strategy; however, note that both strategies have total and partial knowledge about the dynamics of the returns, respectively. As we mentioned previously, the SW strategy, Eq. (6.13), knows the dynamics of the stylized returns. On the other hand, the MLS strategy, Eq. (6.19), knows the periodicity T of the returns, which is used to calculate the best memory size by means of Eq. (7.9). This previous knowledge gives some advantage to these strategies over the GACE strategy, which only needs the specification of $G_{\max}$. We note that the GACE strategy evolves quite quickly, yielding a set of investment strategies with a clear tendency to lead to more gains than losses. This is shown for long-lasting periodicities in particular, where ever increasing budget growth is observed. Interestingly, the GACE strategy performs much better than the reference strategy CP and performs on certain occasions as good as the MLS strategy, particularly for returns with small noise.

8.5. Conclusions

In this chapter, we presented an adaptive investment strategy called *Genetic Algorithm for Changing Environments* (GACE), which is a new approach based on evolution for the cor-

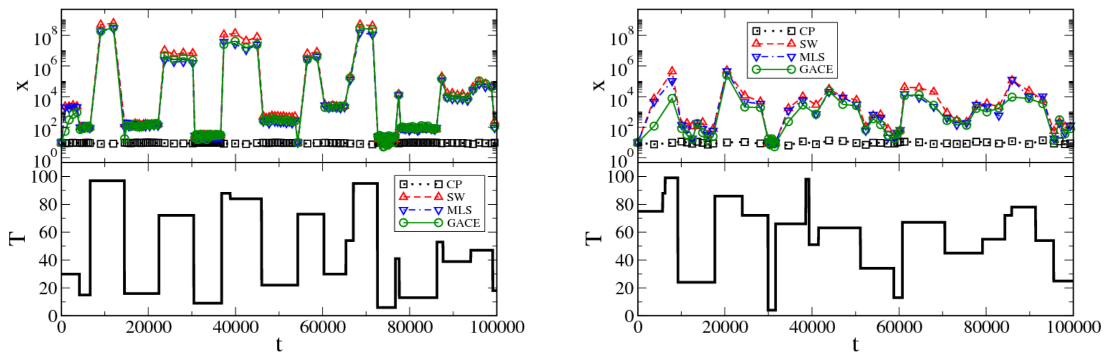


Figure 8.7.: (top) Budget over time for different strategies, assuming $q_{\min} = 0.1$ and $q_{\max} = 1.0$ for all strategies. For MLS, Eq. (7.9) was used to calculate the memory size, M, and for GACE, the parameters shown in Table 8.1. (bottom) Periodicity of the returns over time, Eq. (8.7), with parameters: $T_{\max} = 100$, $t_{\max} = 10^4$, and amplitude noise: (left) $\sigma_2 = 0.1$ and (right) $\sigma_2 = 0.5$.

rect mapping of proportions of investment to patterns that may be present in the returns. We analyzed the performance of GACE for different scenarios, and compared its performance over time to other strategies that were used as a reference. We showed that after a given number of time steps, the GACE strategy reaches a set of investment strategies that can outperform simple strategies like those that invest always a constant proportion of investment. We showed that even though the GACE strategy has no knowledge of the dynamics of the returns, it may lead to large gains, performing as well as other strategies with some knowledge. This is shown for long-lasting periodicities in particular, where ever increasing budget growth was observed. This means that in the presence of long-lasting periodicities, the longer the agent uses the adaptive strategy the larger the profits per cycle.

In this study, we used artificialy generated stylized returns, which are based on a sinusoidal function; however, it can be shown that for other type of periodic functions, the GA would eventually find the best strategy in the same way as for the sinusoidal function. Despite the fact that the GACE strategy proposed in this chapter was mainly used to find a good set of proportions of investment, it is important to note that this strategy can be applied to other type of scenarios - for example, scenarios where the agent has to control other kind of resources, like energy, time consumption, etc.

As mentioned in Section 8.1, most of the investment strategies based on machine learning approaches are based on genetic programming techniques and neural networks and in this chapter a new approach based on the standard genetic algorithm was presented. Moreover, the standard genetic algorithm, when applied to non-stationary environments, is to our knowledge, not yet worked out as it is done in this chapter. The same applies to the extensions proposed in this paper for the genetic operators which allow the genetic algorithm to track non-stationary returns. Another important contribution in this chapter is the use of the investment scenario as a test-bed for different investment strategies and to investigate the adaptability of strategies based on machine learning approaches.

8.6. Further Work and Extensions

Further work includes analyzing of the performance of the GACE strategy for real returns and comparing the performance of GACE with other similar approaches like Genetic Programming techniques, Neural Networks, and Reinforcement Learning. It would also be useful to extend this approach for optimal portfolio diversification, for which a large number of algorithms have been proposed which deal with the research areas of optimization, stochastic simulation and decision theory.

Finally, we note that the proposed adaptive investment strategy may be interesting for the research area of pattern recognition of time series. By making proper changes in the fitness function, a useful algorithm could be obtained for the detection and measurement of periodic signal in time series.

Some other extensions to the approach presented in this chapter are, for example, the use of GAs for multi-modal functions, in which a number of GAs, also called building blocks, act together to find multiple optimal solutions for a given problem. The main goal for this approach is to find the condition of steady-state innovation [Goldberg, 2002]. For this, the relationship between selection and innovation needs to be investigated in order to describe, by means of a control map, those cases for which a steady-state innovation or a premature convergence may be present.

8.6.1. Use of Fourier Techniques to Determine Periodicity Changes

Fourier Series of Square Wave

In Section 6.4.2, it was mentioned that the agent may use an investment strategy based on a rectangle function, where the agent decides to invest given the sign of the last RoI, or given some measure of the noise of the RoIs. Another possibility is to assume that the agent decides to use an investment strategy described by the following square wave function:

$$q_i(t) = \begin{cases} q_{\max} & \text{for} \quad 0 \le t < \frac{\hat{T}}{2} \\ q_{\min} & \text{for} \quad \frac{\hat{T}}{2} \le t < \hat{T} \end{cases} \tag{8.10}$$

where $\hat{T}$ is the estimated periodicity of the RoI. This square wave function can be expressed also using Fourier series of the form:

$$q_i(t) = \sum_{k=-N}^{N} a_k e^{j(2\pi/\hat{T})kt} \tag{8.11}$$

where the Fourier series coefficients, a_k, can be calculated as follows [McClellan et al., 2003]:

$$a_k = \begin{cases} \frac{1}{j\pi k}(q_{max} - q_{min}) & k = \pm 1, \pm 3, \pm 5, .. \\ 0 & k = \pm 2, \pm 4, \pm 6, .. \\ \frac{1}{2}(q_{max} - q_{min}) & k = 0 \end{cases} \tag{8.12}$$

Use of *FFT* and Wavelet *db4* to Determine RoI's Periodicity

Finally, another possible extension would be to consider an investment strategy based on the wavelet *db4*. In this approach, the agent uses the wavelet to determine when there is a change in the periodicity of the RoI and apply the Fast Fourier Transform to determine the periodicity of the actual RoI.

For example, Fig. 8.8 (top) shows some periodic returns with 3 different periods over time. Fig. 8.8 (middle) shows the transformed signal obtained using a wavelet db4. Note that changes in the periodicity are detected by a peak in the signal. Fig. 8.8 (bottom) shows the FFT transformation at $t = 200$. Note that the periodicities $T = 12$ and $T = 86$ are detected; however, a mechanism has to be introduced to reset the saved returns for the FFT in order to consider only those returns after a change in the periodicity is detected.

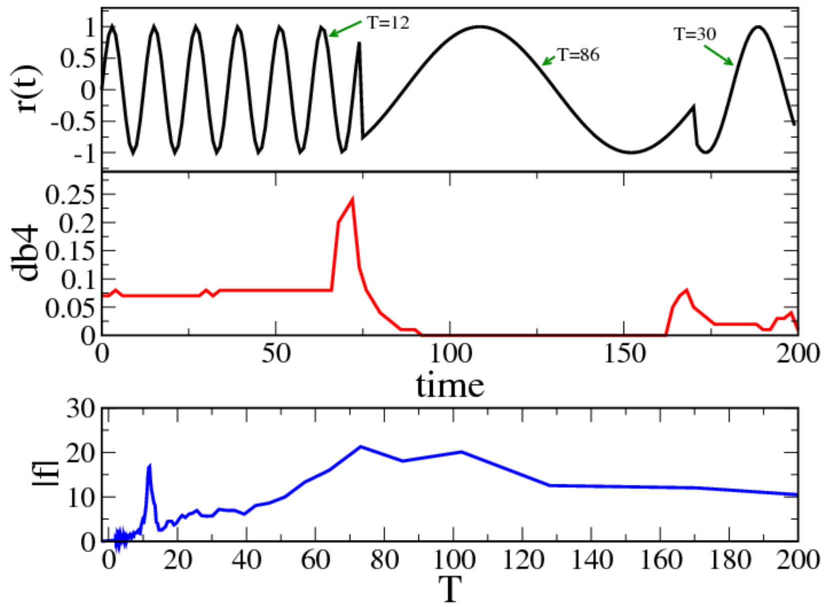


Figure 8.8.: (top) Periodic returns with 3 different periods over time, (middle) wavelet db4 transformation, changes in periodicity are detected by a peak in the signal and (bottom) FFT transformation magnitude vs. period obtained at $t = 200$.

9. Formation of Common Investment Networks

This chapter extends the investment model to include the formation of common investment projects. Each project is conducted by an initiator who tries to convince other agents to invest in his project. The decision of an agent to invest depends on the previous experience with the particular initiator. The dynamics of the budget and some properties of the networks are investigated.

9.1. Introduction

In this chapter we investigate the formation of common investment networks between agents. This topic is usually treated in the area of economic Multi-Agent Systems (MAS) which has recently gained much attention under the label ACE – *agent-based computational economics*. For example, ACE models have been proposed to study the relationship between market structure and worker-employer interaction networks [Tesfatsion, 1998, 2001a]. Recently, some models for the endogenous formation of trade networks have been also proposed (see for example [Albin and Foley, 1992; Tesfatsion, 1997; Vriend, 1995]). A key concern in these studies is the emergence of a trade network among a collection of buyers and sellers who adaptively select their trade partners. Agents perform these selections by looking at their past experiences with these partners. For example, Kirman and Vriend [2001] show an ACE model of the wholesale fish market in Marseilles, where the authors try to understand the buyer loyalty to sellers by means of repeated business. Interestingly, buyers learn to be loyal as sellers learn to offer a higher payoff to loyal buyers.

Apart from studying the dynamics of payoff, loyalty and partner selection, the topology of the networks emerging from simple models, MAS or from real systems has been also investigated by many researchers, see [Albert and Barabási, 2002; Battiston et al., 10; Bornholdt and Schuster, 2002; Dorogovtsev and Mendes, 2003; Newman, 2003; Xulvi-Brunet, 2006]. For the purposes of this PhD thesis, we review in Section 9.4 some of the properties of the networks that are commonly used to characterize their topology.

Thus, in this chapter, we propose an investment model where agents interact with each other in order to establish investment projects. The following two research questions are investigated in this chapter: *how does the interaction between agents influence the evolution of the budget as well as the trust reputation between the agents? How does the topology of the networks is characterized and how does it evolve over time?* To answer these questions, we first present in Section 9.2 the model for the formation of common investment networks; Section 9.3 presents some simulation results for the analysis of the evolution of budget, trust and reputation between the agents; Section 9.4 investigates the formation of networks for different parameter values in the model; and finally in Section 9.6 some extensions for these investigations are proposed.

9.2. The Model

Following the investment model presented in Eq. 2.9, consider now a *multi-agent system* of N agents, where each agent posses a *budget* $x_k(t)$ that evolves over time given the following dynamic [Navarro and Schweitzer, 2003]:

$$x_k(t+1) = x_k(t)\Big[1 + r_{mk}(t)\, q_k(t)\Big] + a, \tag{9.1}$$

where $r_{mk}(t)$ denotes the return on investment (RoI) that the agent k receives from its investment $q_k(t)$ in project m. Note that $q_k(t)$ denotes a *proportion of investment*, i.e the fraction or ratio of the whole budget of agent k. Note also that each agent starts with the same initial budget $x(0)$.

Now, in order to launch a particular investment project m at time t, a certain minimum amount of money I_{thr} needs to be collected among the agents. The existence of the investment threshold I_{thr} is included here in order to enforce the interaction between agents, as they need to *collaborate* until the following condition is reached:

$$I_m(t) = \sum_{k}^{N_m} q_k(t)\, x_k(t) \geq I_{\text{thr}}, \tag{9.2}$$

where N_m is the number of agents collaborating in the particular investment project m. Note that if I_{thr} is a fixed number, then a small number of "wealthy" agents can easily overspend the threshold. There may be different investment projects m at the same time, but at the moment it is assumed that each agent participates in only one investment project at a time.

The first essential feature to be noticed for the formation of common investment networks is the establishment of preferences between agents. It is assumed that the decision of an agent to collaborate in a project will mainly depend on the previous history it has gained with other agents. Consider an agent k which accepts to collaborate in the common investment project m initiated by agent j. Thus, agent k receives the following payoff at time t:

$$p_{kj}(t) = x_k(t)\, q_k(t)\, r_m(t). \tag{9.3}$$

Reiterated interactions between agent k and agent j lead to different payoffs over time that are saved in a decision weight:

$$w_{kj}(t+1) = p_{kj}(t) + w_{kj}(t)\, e^{-\gamma}, \tag{9.4}$$

where γ represents the memory of the agent and the initial decision weight is zero, $w_{kj}(0) = 0$.

The payoffs obtained from previous time steps t may have resulted from the collaborative action of different agents,however, these are unknown to agent k, i.e agent k only realizes the initiator of the project, agent j. Furthermore, in order to mirror reality more closely, it is assumed that there are more investors than initiators of projects. For this, we consider that from the population of N agents only a small number J are initiators, i.e. $J \ll N$. Note that the reputation of an initiator j can be calculated as follows:

$$W_j(t+1) = \sum_{k=0}^{N} w_{kj}(t). \tag{9.5}$$

In the same manner, the trust of an agent k to an initiator j can be calculated as follows:

$$W_k(t+1) = \sum_{j=0}^{J} w_{kj}(t). \tag{9.6}$$

For more complex trust and reputation models see [Sabater and Sierra, 2005; Walter et al., 2008].

In general, the investment model proceeds as follows: At every time step t an initiator is chosen at random from the population and an investment project is assigned to him. The initiator randomly tries to convince other agents to invest in his project until it has collected at least the threshold amount I_{thr}. For this, we use a Gibbs or Boltzmann distribution to determine the probability that the contacted agent k may accept the offer of agent j:

$$\tau_{kj}(t) = \frac{e^{\beta w_{kj}(t)}}{\sum_{i=1}^{J} e^{\beta w_{ki}(t)}}, \tag{9.7}$$

where in terms of the weight w_{kj}, the probability $\tau_{kj}(t)$ considers the good or bad previous experience with agent j with respect to the experience obtained with other initiators; and β denotes the greediness of the agent, i.e. how much importance does the agent give to the decision weight w_{kj}. In order to take a decision, agent k uses a technique analogous to a roulette wheel where each slice is proportional in size to the probability value $\tau_{kj}(t)$. Thus, agent k draws a random number in the interval $(0, 1)$ and accepts to invest in the project of agent j if the segment of agent j in the roulette spans the random number. Finally, an initiator j stops to contact other agents if either the investment project has reached the threshold I_{thr} or if all agents in the population have been asked for collaboration. In order to keep the multiplicative-additive dynamics from the investment model studied in previous chapters, only the initiator and investors of a launched project receive an external income. Also note that the term β is typically called *temperature* and in this case for small β values (for example $\beta = 0.001$), initiators are equally probable to be chosen. In terms of the behavior of the agent this means that the agent decides to explore more the payoffs that the initiators can offer by not taking into account the previous obtained payoffs. For large β values (for example $\beta = 1$), previous experience with the initiators is enhanced, i.e. the agent decides to explode more those initiators supplying the largest positive payoffs. Finally, note that the initiator will always invest in his own project, i.e. he neither looks at his own performance nor compares it with his experience with other initiators.

Now, if the project could be launched, then it has to be evaluated. The evaluation should in general involve certain "economic" criteria that also reflects the nature of the project. However, we do not want to include such assumptions and for simplicity we assume that the failure or success of an investment project I_m is randomly drawn from a uniform distribution, i.e. $r(t) \sim U(-1, 1)$. A more realistic assumption would include also gains with $r \gg 1$, while the loss is still bound to the maximum investment value.

9.3. Results of Computer Simulations

In this section we investigate the dynamics of the investment model. In general, the model has the following parameters: N the number of agents, J the number of initiators, I_{thr} the investment threshold, $x_k(0)$ the initial budget of agent k, $q_k(t)$ the proportion of investment of agent k, γ the memory and β the greediness. For simplicity, we assume that the initial budget is the same for all agents, i.e $x_k(0) = x(0)$, and the number of time steps t refer only to established projects. Moreover, the proportion of investment is assumed to be constant and the same for all agents i.e. $q_k(t) = q = const$. We start our analysis by investigating the dynamics of the evolution of the budget of the agents, the number of investors and the reputation of initiators over time for different proportion of investment q. For this, we consider the parameter values in Table 9.1 for the simulation experiments.

Table 9.1.: Parameter values of the computer experiments for the investment networks formation model.

Parameter	Value
Number of agents	$N = 10000$
Number of initiators	$J = 100$
Number of time steps	$t = 100000$
Investment threshold	$I_{\text{thr}} = 9$
Return on Investment (RoI)	$r \sim U(-1, 1)$
Initial budget of the agent	$x(0) = 1$
Income of the agent	$a = 0.5$
Memory of the agent	$\gamma = 0.1$
Greediness of the agent	$\beta = 1$

In the first computer experiment, the investment model was simulated for the parameter values in Table 9.1 and for different constant proportion of investment q. Fig. 9.1 (left) shows the evolution of the budget distribution over time for $q = 0.5$. It can be seen that the probability distribution of the budget has a power law in the tail, a property of investment models based on multiplicative processes repelled from zero, discussed previously in Section 2.3. Note that the distribution of the budget for the investment model presented in this Chapter follows the same distribution as for the investment model in Section 2.2. This means that in the long run the distribution of the budget can be described using the stationary solution in Eq. (3.26). Moreover, it can be seen that the most probable budget value x_{mp} agrees with the analytical result in Eq. (4.13) with $x_{\text{mp}} \approx 50$. However, note that a larger number of time steps (established projects) are needed to reach convergence, because now in this model the agents are not always performing investments. This is the case when agents having a small fixed proportion of investment are not able to invest together more than the investment threshold I_{thr} needed to establish a project. This is shown in Fig. 9.1 (right), where the distribution of the budget is shown at time step $t = 100000$ for different proportion of investment q. Note that even for a large number of time steps, the budget distribution for agents with a proportion of investment of $q = 0.1$ has not yet converged to a stationary distribution, whereas for $q = 0.5$ and $q = 0.9$, the distribution reached a stationary state after $t = 70000$ and $t = 50000$ time steps respectively.

Furthermore, in order to understand the role that initiators play in the dynamics of the investment model, we examine the evolution of their budget and reputation over time. In the experiments run previously a number of initiators of $J = 100$ was assumed. Note that for a small number of initiators it is better, for the sake of clarity, to show the rank-size distribution of the budget instead of the probability distribution of the budget. Later on, a larger number of initiators is considered, which allows for appropriate visual representations of the budget probability distributions of the initiators. Fig. 9.2 (left) shows the rank-size distribution of the budget of the initiators, it can be seen that the slope of the distribution

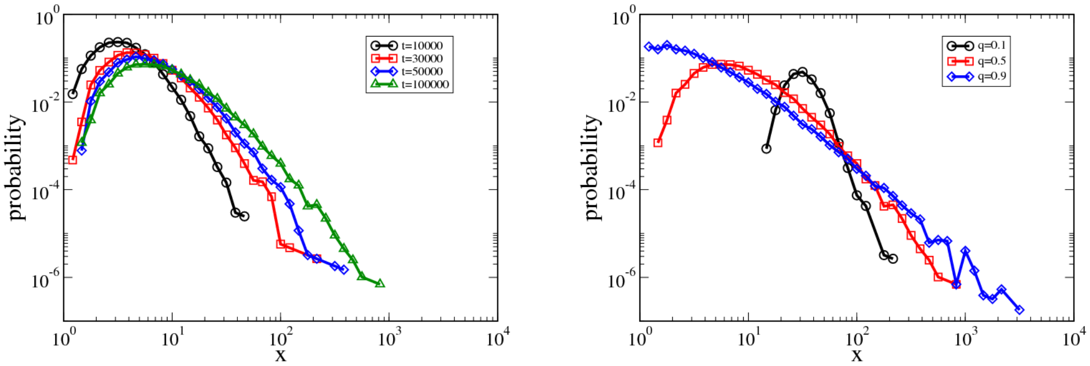


Figure 9.1.: (left) Evolution of the budget distribution over time for $q = 0.5$; (right) budget distribution at time step $t = 100000$ for different proportion of investment q. For parameter values as in Table 9.1.

increases over time. It is also interesting to examine the evolution of the budget of the initiator with the largest and the smallest budget at the end of the simulation. This is shown in the inset of Fig. 9.2 (left), where it can be seen that the budget of the best agent was not always increasing over time. We note that, it would be interesting to show the influence of successful and non-successful projects at the beginning of the simulation on the evolution of the budget over time. However, because of the fact that initiators randomly ask other agents to invest in their project, they do not really have a preference over wealthy or non-wealthy agents, therefore, it is the reputation of the initiators the property of the initiator that plays a more important role in the dynamics of the investment model rather than its budget. Fig. 9.2 (right) shows the rank-size distribution of the reputation of the initiators, Eq. (9.5). It can be seen that the distribution does not change over time, and only for a small number of agents there is a significant increment or decrement of the reputation over time. Moreover, it can be shown that the average value of the reputation has a shift to larger positive values over the course of time. This occurs because of the external incomes a in Eq. (9.1), which are implicitly included by the term $x_k(t)$ in the dynamics of the decision weights in Eq. (9.4). Moreover, the inset in Fig. 9.2 (right) shows that the reputation of the best and the worst initiator (in terms of reputation), which indicates the presence of no symmetrical positive/negative reputation values.

Now, in order to understand the influence of the rest of the parameters in the dynamic of the model, the proportion of investment is fixed to $q = 0.5$ and some computer experiments are run for different number of initiators J. An interesting parameter in the dynamics of the investment model is the number of initiators J. Note that if a less number of initiators is considered, then more investors will be willing to invest in their project, which means that a larger amount of investment can be collected by the initiators. In order to let a large number of agents to invest in a project from the very beginning of the simulation, the investment threshold I_{thr} has to be proportional to the initial amount of money that the initiators may be able to collect. Because of this the following investment thresholds were assumed for the corresponding number of initiators: $I_{\text{thr}} = 490$ for $J = 10$; $I_{\text{thr}} = 9$ for $J = 100$; and $I_{\text{thr}} = 4.5$ for $J = 1000$. Fig. 9.3 (left) shows the budget distribution of the agents in the population for the previous mentioned number of initiators J and investment thresholds

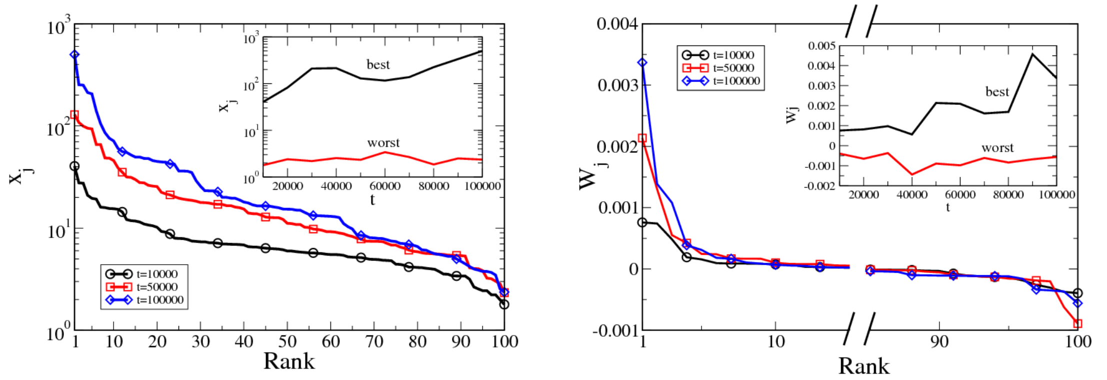


Figure 9.2.: Evolution of the rank-size distribution of initiators for: (left) budget; (right) reputation. Insets show the reputation of the best and the worst initiator. Parameter values as in Table 9.1.

$I_{\rm thr}$. Note that the tail of the distribution has a power law distribution and the larger J the larger the slope of the power law. The reason for this is that a small number of initiators collect more money from the investors leading to larger profits and looses which over time lead to wider distributions than for a large number of initiators. Moreover, Fig. 9.3 (right) shows the budget distribution of the investors and the initiators separately. It can be seen that the initiators tend to accumulate more budget than the rest of the agents This occurs because by definition, initiators invest more frequently than those that are no-initiators.

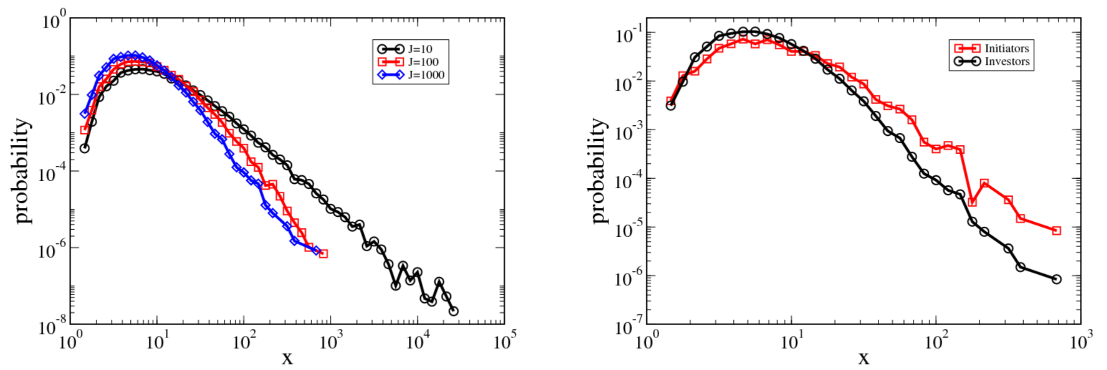


Figure 9.3.: Budget distribution for: (left) all agents and for different number of initiators J; (right) for investors and initiators separately for $J = 1000$. For proportion of investment $q = 0.5$ and further parameter values as in Table 9.1.

Moreover, Fig. 9.4 shows the ranked size distribution of the reputation, Eq. (9.5), of the initiators and the trust, Eq. (9.6), of the investors to the initiators. It can be seen that both distributions have a similar form, however, we note that there is a small number of agents with relative high or low reputation and trust values. This is due to the fact that wealthy agents modify their decision weights much more than less wealthy agents. This larger positive or negative weights play an important role in the establishment of projects, however, because of the fact that agents are contacted at random less wealthy agents are still able to participate in projects. This can be seen from the budget distribution of the agents which has the typical power-law behavior from other investment models.

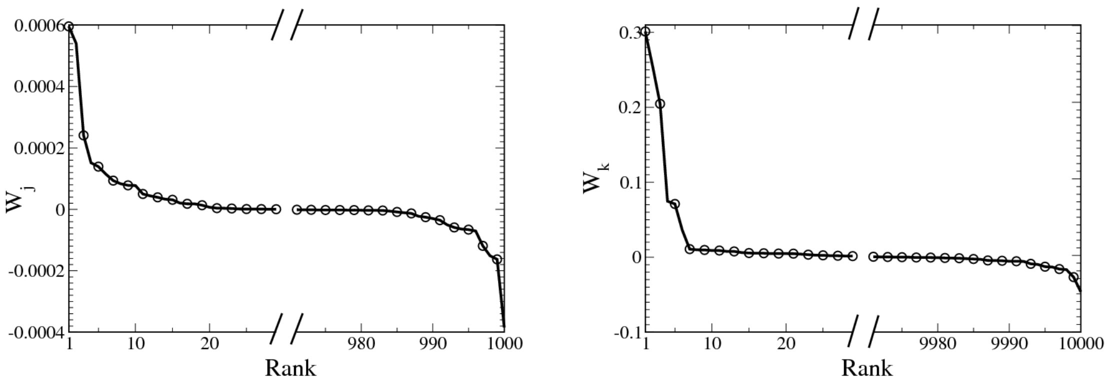


Figure 9.4.: Ranked size distribution at time step $t = 100000$ for: (left) reputation of initiators, Eq. (9.5), and (right) trust of investors, Eq. (9.6). For proportion of investment $q = 0.5$, number of initiators $J = 1000$ and further parameter values as in Table 9.1.

9.4. Structure of Common Investment Networks

In this section the structure of the networks is analyzed. As we did in the previous section, we start our investigations analyzing the topology of the networks for different constant proportion of investment $q(t) = const$. For this, we run different computer experiments for a small population of agents $N = 1000$ (N is also the number of nodes in the network) and the other parameter values as in Table 9.1. The first experiment investigates the influence of the proportion of investment in the properties of the network. Fig. 9.5 shows the results for different proportion of investment q at time step $t = 1000$. Note that these networks have two types of nodes, where the bold nodes are investors and the gray nodes are initiators. Based on visual impression, the density of the network decreases with respect to the proportion of investment. This occurs because agents investing more also tend to loose more, which leads to more mistrust.

However, from the visual representation of the network it is not possible to draw many conclusions from the dynamics of the networks. Because of this, the following properties of the network are obtained:

- Number of links V and maximal degree $k_{\max}$ (the degree of the highest degree vertex of the network).

- Average path length l: the average shortest distance between any pair of nodes in the network, i.e. the average of the minimum number of links that are needed to cross from one node to another node. The measure became important after the investigations of Milgram [1967] who demonstrated in a series of interesting experiments that the spread of information from person to person verifies what it is now known as small-world behavior.

- Clustering coefficient C: measures the transitivity of the network [Watts and Strogatz, 1998]. Based on the local value:

$$C_i = \frac{\text{number of triangles connected to node } i}{\text{number of triples centered on node } i}, \tag{9.8}$$

the mean clustering coefficient of the network is defined as:

$$C = \frac{1}{N} \sum_i C_i. \tag{9.9}$$

This property is important as it has been shown by Watts and Strogatz [1998] that in real networks the clustering coefficient is usually much larger than the clustering coefficient in a random network with the same number of nodes and links.

- Degree distribution $P(k)$: the probability that a randomly selected node of the network has exactly k links, i.e. the fraction of nodes in the network that have degree k. It has been shown that the degree distribution of many real networks have a power law in the tail [Albert and Barabási, 2002; Albert et al., 1999]:

$$P(k) \sim k^{-\alpha}, \tag{9.10}$$

for some constant exponent α. Networks with a power-law distribution are also called scale-free networks and have been investigated by many researchers [Albert and Barabási, 2002; Dorogovtsev and Mendes, 2003; Ebel et al., 2003; Strogatz, 2001].

In order to understand better the dynamics of the networks that emerge from the interaction of the agents, some of the previous listed properties and measures are obtained on the following.

Fig. 9.6 (left) shows the number of links V over time for different proportion of investment q. Note that the simulation results fit a power law, where the slope decreases if the proportion of investment increases. As expected, the number of links increases much faster over time for larger proportion of investment q, however, note that the number of links for small q is much larger in the beginning of the simulations than for larger q. This occurs because in the beginning the decision weights are zero and their values are much more modified by the payoffs than by the previous weight. Note also that after a large number of time steps, the number of links for $q = 0.5$ and $q = 0.9$ are of the same size and based on visual impression, after a large number of time steps, the larger q, the larger the number of links in the network.

To gain more insight into the dynamics of this model, Fig. 9.6 (right) shows the maximal degree k_{max} over time for different proportion of investment q. It can be seen that the maximal degree of the network also increases over time with a power-law behavior. Based on visual impression, after $t = 10^5$ time steps the number of links and the maximal degree have a similar value for both proportions of investment $q = 0.5$ and $q = 0.9$. This occurs mainly because in the long run, the decision weights of some agents are largely modified by the previous decision weights than by their current payoff.

Fig. 9.7 (left) shows the evolution of the clustering coefficient C over time for $q = 0.5$. It can be seen that the clustering coefficient increases for small proportions of investment.

This occurs basically because of the small number of initiators in the system. Note again that a small proportion of investment leads to a higher clustering in the network because of the mistrust that large losses generate in the investors. Note that the evolution of V, $k_{\max}$ and C over time can be described by a power-law. Thus, an approach like the one used in Section 4.3, could be used to obtain a scaling function for these measures with respect to the number of time steps and the proportion of investment; however, this is beyond the scope of this chapter and is left for further work.

Fig. 9.7 (right) shows the evolution of the degree distribution over time for $q = 0.5$. Note that two degree distributions emerge from our model. The first one has small k values and corresponds to the links of the non-initiators. The second one has large k values, which, for visibility reasons, is shown in the inset of Fig. 9.7 (right) and corresponds to the initiators. It can be seen that for both cases, the degree distribution follows a binomial distribution. First studied by Erdös and Rényi [1959], random graphs show the property of binomial degree distribution, which for large number of nodes can be good approximated by a Poisson distribution. Thus, these experiments suggest that the investment networks here obtained are of the type of a random graph. On the other hand, it has been shown that many real networks present a "small world" property, which in general means that the network has the following properties:

1. Relative high clustering coefficient, larger than the clustering from a random network with the same degree, i.e. $C \gg C_{\text{rand}}$.

2. Small average path length which scales logarithmically with the size of the network, i.e. $l \sim \log N$.

3. Degree distribution with a power-law behavior in the tail.

Table 9.2 shows some of the most important characteristics for different number of investors N and initiators J for a large number of time steps, i.e. $t = 100000$. For each network we indicate the average degree $\langle k \rangle$ (the first moment of the degree distribution), the average path length l and the clustering coefficient C. For comparison reasons we include the average path length $l_{\text{rand}} = \log(N)/\log(\langle k \rangle)$ and the clustering coefficient $C_{\text{rand}} = \langle k \rangle / N$ that can be obtained from a random network with the same average degree $\langle k \rangle$ of the investment networks.

Table 9.2.: Properties of the investment networks for different number of investors and initiators. For each network the properties measured are: the average degree $\langle k \rangle$, the average path length l and the clustering coefficient C. For proportion of investment $q = 0.5$, $t = 100000$ and further parameters as in Table 9.1.

N	J	V	$k_{\max}$	$\langle k \rangle$	l	C	l_{rand}	C_{rand}
1000	10	4847	517	0.9694	2.05766	0.74557	-	0.0009694
2000	20	19972	1050	3.9944	1.99365	0.71337	5.488	0.0019972
3000	30	41073	1475	8.2146	1.99314	0.71130	3.8018	0.0027382
10000	100	134279	1477	26.86	2.1563	0.24136	2.7989	0.002686

It can be seen that the average degree $\langle k \rangle$ increases with respect to the system size. This is no surprise since according to Fig 9.7 (right), the degree distribution shifts to larger positive values over time. Note that for the parameters: $N = 1000; I = 10$, the average degree path of the network is less than one, which means that the network has either trees or clusters containing exactly one link. On the other hand, for the rest of the parameter values in Table 9.2, it can be seen that the networks show a small average path length $l \approx 2$, meaning that any investor or initiator in the network is connected to each other, through two links in average. It can be seen that for large number of nodes, the average path of the networks is approximately equal to that from a random graph generated with same average degree of the investment network. It can be seen from the Table 9.2 that the clustering coefficient of the investment networks is larger than the clustering coefficient that may be obtained from a random network, which indicates the presence of transitivity in our networks. This occurs mainly because of the large number of investors connected to initiators. Note that the values of C in our networks are similar to the clustering coefficient obtained for real bipartite networks, for example, [Watts and Strogatz, 1998] report that the clustering coefficient for the network of movie actors is $C = 0.79$. Note that a property of random networks is that the clustering coefficient decreases with respect to the size of the network. We see from the Table 9.2, that in our networks, the clustering coefficient also decreases with respect to N.

9.5. Conclusions

The influence of the parameters in the dynamics of the budget and the dynamics of the investment networks were investigated. It is shown that the budget of the agents reaches a stationary distribution after some time steps and presents a power law distribution on the tail, property extensively discussed in the first part of this PhD thesis. The topology of the investment networks emerging from the model were analyzed showing that the networks present some of the typical characteristics of real-life networks like a high clustering coefficient and short average path length. However, it is shown that the degree distribution of the investment networks does not follow a power-law behavior, which is usually found in real-world networks, but rather a binomial distribution which is more related to random networks.

We have mainly focus our investigations on the feedback describing the establishment and reinforcement of relations among agents and initiators, which dynamic is mainly driven by the decision weights $w_{kj}(t)$, Eq. (9.4). This is considered a “social component” of the agents’ interaction and it was shown how this feedback process based on positive or negative experience may lead to the establishment of networks among agents. For simplicity, we have just assumed a random selection of failure or success, but we note that more elaborated economic assumptions, such as market dynamics based on supply and demand, can be taken into account as well.

The investigations in this chapter showed results mainly for a fixed number of agents $N = 10000$, we note that an analysis of the network order is needed, i.e. to analyze the properties of the networks that emerge from simulations with different number of agents.

We note also that agent’s external incomes influence the ever increasing reputation and trust of only a few number of agents. This occurs because the dynamics of the decision weights of wealthy agents are mainly influenced by their current payoff, Eq. (9.4). These results indicates that an extra mechanism or behavioral component needs to be added to the model in order to obtain networks with a stationary power-law degree distribution, property which is usually found in real-world networks.

9.6. Further work

As we noted previously, because of the fact that two different type of nodes are present in the system, the networks that emerge from these experiments are of the type "bipartite graph". These types of networks are also studied by converting them into one-mode networks, this transformation is done by considering that two nodes of the same type are connected when they are connected to a node of different type [Wasserman and Faust, 1994].

We note also that further experiments are needed for different memory γ and greediness β values to understand the influence of these parameters in the dynamics of the networks.

It would be interesting to analyze the role of the memory γ and greediness β which describe the exponential decay of the past experience and the importance of the weights $w_{kj}(t)$ in the decision process respectively. For example, without memory, i.e. $\gamma \to \infty$, it is expected that agents just randomly gather for a certain project which may describe the *random* scenario. On the other hand, if agents' memory is too long, i.e. $\gamma \to 0$, it is expected that any positive and negative experience will last forever and changes in the structure of the networks may hardly be observed, which may describe the *frozen* scenario.

As it was done in Chapter 7 a behavioral component could be also included in order to account for the failure or success of the previous investments and to modify the investment proportion $q(t)$ of the agent accordingly. Another extension would be to include costs, for example when asking other agents for investment (for the initiators) and for attending requests (for the investors). For this, initiators need to incorporate an extra mechanism to be aware of the advantage of asking those agents that they "believe" may accept to invest in his project and not at random as it was done in our experiments. On the other hand, investors need to somehow pay more attention to those initiator that they "believe" may have successful projects, i.e. to be more or less greedy depending on the experience with a particular initiator.

It would be also interesting to change the multiplicative additive random approach of the investment model to a multiplicative entry/exit process, see Section 2.3.4. Note also that using the approach of utility theory in Appendix 10.2, the investment model can be extended to a model where agents may be willing to pay/receive money in order to persuade/accept investing in a common investment project.

Another possible extension is to consider that the agent invests at the same time in different investment instruments. This means that, in this case an agent has now the opportunity to invest in one or more investment options.

In the literature, the problem for portfolio optimization has been handled by many researchers from economics and finance [Cover, 1991; Kahnemann and Riepe, 1998; Markowitz, 1952, 1991]. Note that some of the problems related to optimal diversification of resources can be investigated using the following extension of the investment model presented in Eq. 2.9:

$$x(t+1) = x(t)\left[1 + \sum_{j=1}^{N_m} r_j(t)\, q_j(t)\right] + a, \tag{9.11}$$

where now $q_j(t)$ stands for a diversified investment proportion vector and $r_m(t)$ corresponds to the return obtained from each investment option m. This means that the agent invests at time step t a portion $I(t)$ of its total budget:

$$I(t) = \sum_{m=1}^{N_m} q_m\, x. \tag{9.12}$$

Since q_m represents always a portion of the total budget x invested in the market r_m, it's bound to $q_m \in [0,1]$ and to the following constraint:

$$\sum_{m=1}^{N_m} q_m <= 1. \tag{9.13}$$

Finally, if the returns present correlations in time, i.e. are predictable as those presented in Section 6.3 and Section 8.4.1, the proportions of investment may change according to the portfolio selection strategy. Similar to the evolutionary approach conceived in Section 8.3, a genetic algorithm can be considered, in which each chromosome describes the different portfolio selection possibilities as it is depicted in Table 9.3

Table 9.3.: Chromosome description for proportional investment portfolio optimization.

$chromosom_1 =$	$q_{1,1}$	$q_{2,1}$	...	$q_{m,1}$
$chromosom_2 =$	$q_{1,2}$	$q_{2,2}$	...	$q_{m,2}$
...				
$chromosom_n =$	$q_{1,n}$	$q_{2,n}$	...	$q_{m,n}$

For this case, each $q_{i,j} \in (0,1)$ represents the investment proportion of chromosome j for the investment option i. Once the investment is performed, the fitness of the population can be calculated, given the return $r_i(t)$ for each investment option:

$$f_j(t) = \sum_{i=1}^{m} q_{j,i}\, r_i(t), \tag{9.14}$$

where $f_j(t)$ is the fitness of the chromosome j at time step t.

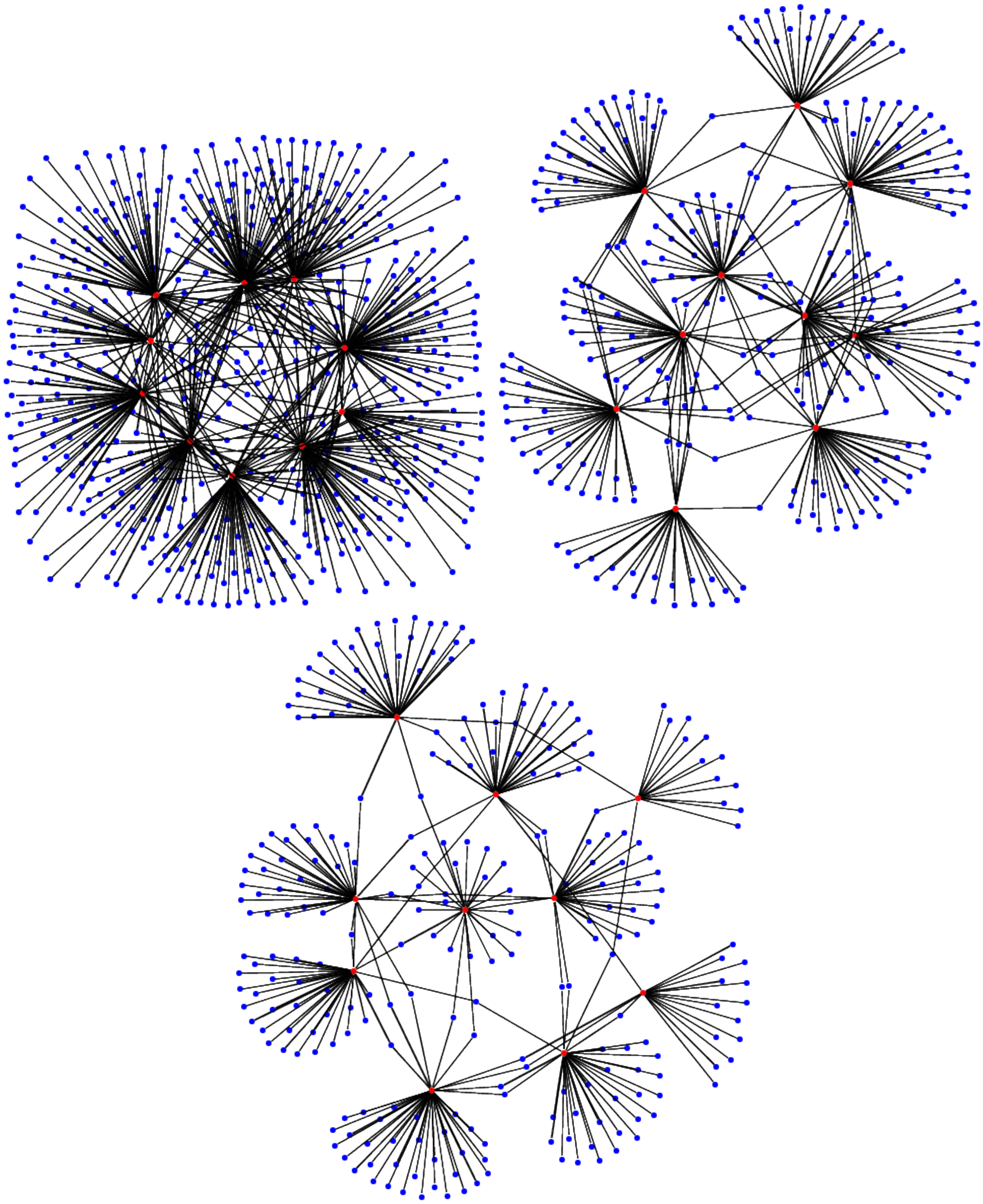


Figure 9.5.: Common investment networks for different proportions of investment at time step $t = 1000$: (left) $q = 0.1$, (right) $q = 0.5$ and (bottom) $q = 0.9$. A link between agents represents a positive decision weight, i.e $w_{kj} > 0$. For $N = 1000$ agents and parameters as in Table 9.1.

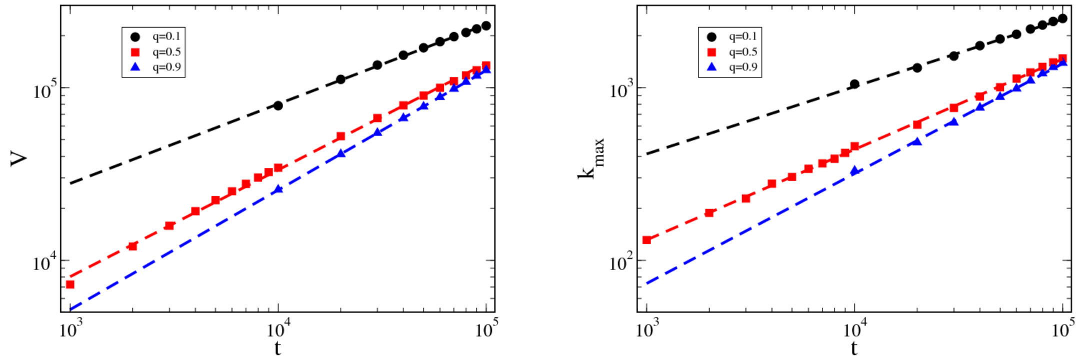


Figure 9.6.: Time dependency of (left) the number of links V and (right) the maximal degree $k_{\max}$ for different proportions of investment q. The dashed lines show the power-law scaling relations for V and $k_{\max}$ with respect of time. Further parameters as in Table 9.1.

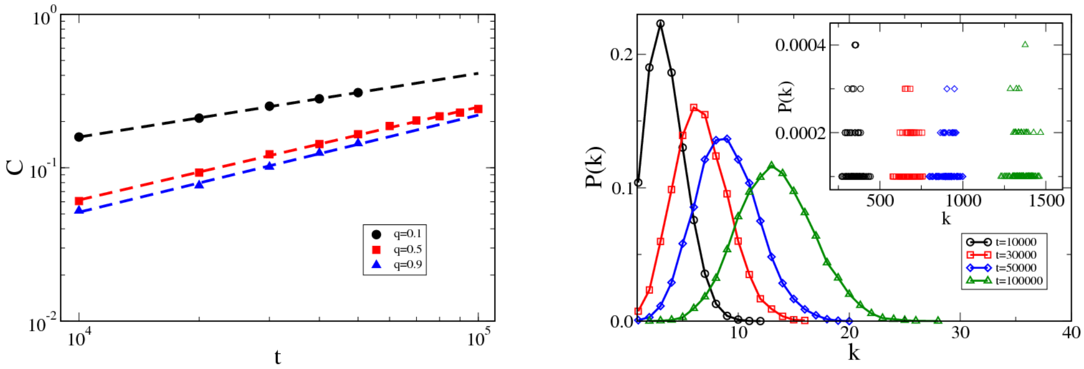


Figure 9.7.: (left) Time dependency of the clustering coefficient C, Eq. (9.9) for different proportions of investment q. The dashed lines (left) represent the scaling relations for C. (right) Time dependence of the degree distribution $P(k)$ for proportion of investment $q = 0.5$. The inset shows the degree distributions for the initiators. Further parameters as in Table 9.1.

10. Concluding Remarks

This Chapter presents the main contributions, conclusions and the future extensions for the investigations in this PhD thesis.

10.1. Main Contributions

The main goal of this PhD thesis was to investigate the properties, behavioral rules and learning mechanisms that lead an agent to obtain larger profits in random and periodic investment environments. In order to achieve this goal, different investment strategies were investigated, for which the function of the strategy depended on the available information from the environment and the processing capabilities (internal architecture) of the agent. For the sake of completeness, a summary of the main contributions is presented in the following.

10.1.1. Investment Model and Fixed Investment Strategies

In Section 2.2, an investment model was presented for which the decisions of the agent (the investor) do not affect the environment [Navarro and Schweitzer, 2003]. In Section 2.3, it was shown that this simple approach can be related to theoretical results for multiplicative stochastic process repelled from zero, which allows for analytical analysis of the investment dynamics.

In Section 3.3, an analytical expression was derived for the stationary probability distribution of an investor's budget when investing constant proportions of budget in a random environment, Eq. (3.26). In Section 4.4, it was shown that the most probable value of the investor's budget scales with the constant proportion of investment, the incomes and the returns, Eq. (4.8). This result was confirmed both by analytical investigations and extensive computer simulations for different stochastic processes [Navarro-Barrientos et al., 2008b].

10.1.2. Investment Strategies and Formation of Common Investment Networks

In Chapter 6, different investment strategies were presented. These can be applied by agents in an investment market scenario with periodic returns and different types and levels of noise. Their performance was compared and analysed in Chapter 7, and it was determined that the strategy based evolution, as described in Section 6.6.2,, outperformed the other strategies in almost all scenarios. The reason for this is that this strategy discovers the principle of always investing large amounts of money when the expected return is positive and avoiding investment when the expected return is negative. It was also shown that the best rule for investment was a *daring behavior* of always investing the complete budget, which outperformed the more intuitive *cautious behavior* of investing an amount proportional to the expected return [Navarro-Barrientos et al., 2008a].

Another contribution from this PhD thesis is the adaptive investment strategy proposed in Section 8.3, also called *(GACE) Genetic Algorithm for Changing Environments*, which is a new approach based on evolution for the correct mapping of proportion of investment to patterns present in the time series of the returns. The performance of GACE was analysed for different scenarios and compared against other strategies in Section 8.4, showing that even though GACE has no knowledge of the dynamics of the returns, after a given number of time steps it may lead to large gains, performing as well as other strategies with some knowledge about the environment [Navarro-Barrientos, 2008a,b].

Finally, in Chapter 9, a model for the formation of common investment networks was presented [Navarro and Schweitzer, 2003]. The influence of the parameters in the dynamics of the budget and the dynamics of the investment networks were investigated. It is shown that the budget of the agents reaches a stationary distribution after a certain number of time steps and presents a power law distribution on the tail, a property discussed in more detail in the first part of this PhD thesis. The investment networks emerging from the model were analysed, showing that, the networks present some of the typical characteristics of real-life networks, such as a high clustering coefficient and short path length. However, it is also shown that the degree distribution of the investment networks does not follow a power-law behavior, which is usually found in real-world networks, but rather a binomial distribution, which is more often associated with random networks.

10.2. Conclusions and Further Work

Recalling that one of the main goals of this PhD thesis was to find the extent to which the internal complexity of agents influences their overall performance, it can be stated that for scenarios with random returns, the agents with a simple architecture outperform those with a complex architecture. On the other hand, for scenarios where returns present periodicities or correlations over time, the agents with a complex architecture outperform the agents with a simple architecture. The major focus on this PhD thesis was more related to issues of computer science, however the results presented here can be also applied for scenarios where an agent needs to optimize the way it manages its energy, resources, expected life time, etc. Note that for the application of these results in an economic context, in particular to financial markets, the results presented and discussed here can be useful for discerning the performance of different *trading* strategies used in noisy market returns. On the other hand, it is important to notice that the different investment models treated in this PhD thesis assume that the actions of the agents have no effect upon the market and consequently, the price of an asset and the return on investment are treated as exogenous variables. This approach can be used to avoid complex assumptions when interaction between agents is considered, i.e. by keeping the dynamic of the return on investment independent of the investment of the agent, the model is analytically tractable and allows for an easier comparison. For example, the analytical derivation of the stationary probability distribution of the budget, Eq. (3.26), and the scaling law function between the most probable budget value and the parameters characterizing the stochastic dynamics, Eq. (4.8). However, note that one of the underlying assumptions of the investment model is that the RoI is constraint to values between $(-1, +1)$, where -1 means a complete loss of the investment and thus is a reasonable lower bound and +1 means a doubling of the investment, chosen for reasons of better tractability. These constraints for $r(t)$ used during the computer simulations may indeed result in deviations from the theoretical prediction if the underlying stochastic process for $r(t)$ frequently gives values outside the interval $(-1, +1)$, which then need to be discarded. This is the case for returns drawn from a normal distribution or modeled using an ARCH or a GACH process. In fact, as these values increase, the agreement between the "truncated" computer simulations and the theoretical approximation based on the full range of r values decreases. Nevertheless, this argument does not restrict the value of the scaling obtained in Eq. (4.8), which is still valid if the analytical prediction is improved by dealing with *truncated* distributions which affect the calculation of $\langle r^2 \rangle$.

An interesting extension for the investment dynamics in Eq. (2.9) would be to extend it to a portfolio scenario where both return $r(t)$ and proportion of investment $q(t)$ become multidimensional variables. This would allow different investment strategies for different assets as pictured in Eq. (9.11). It would be interesting to analyse the dynamics of the budget for this more complex investment model which approximates reality more closely, for example in financial markets, where multi-asset investments and portfolio strategies play the most crucial role [Elton et al., 2003]. Multi-asset optimal investment strategies for risky assets were already discussed 50 years ago, with an interesting relation to gambling [Breiman, 1960]. More recently, investment strategies to readjust portfolios [Merton, 1990] have been extended [Maslov and Zhang, 1998] for a general distribution of return per capital. Similar to the investment models presented in this PhD thesis, these contributions consider exogenous returns which are drawn from a probability distribution or are modeled by stochastic processes.

Despite the simplicity of the investment model presented in Section 2.2, this is not restricted to artificial scenarios only. In fact, the dynamics of the RoI, $r(t)$, can be taken from real time series instead of being modeled by a stochastic process as shown in Chapter 5 where real returns from the stock market were considered in the investment model for fixed investment strategies. However, the most challenging application is in the dynamics of the variable $q(t)$ which describes the agent's decision-making process about the portion of the budget to be invested. Any realistic investment scenario deals with the problem of how to correctly adjust the proportion of investment $q(t)$ over time based on the observation of previous returns $r(t)$. For this, different investment strategies were presented and their performance was compared for artificial generated stylized returns based on a sinusoidal function. It was shown that an evolutionary approach based on a Genetic Algorithm (GA) outperformed the other strategies for different scenarios. However, it can be shown that for other type of periodic functions, the GA will eventually find the most proper strategy in the same way as for the sinusoidal function. Thus, it is important to notice that the investment models presented in this PhD are also suitable to be used as test-beds for investment strategies, assisting the comparison, analysis and understanding of the adaptability and learning capabilities of strategies.

Regarding the behavior of the agent, another important extension would be to improve the (representative) agent behavior by not only allowing decisions to be made based on the expected return (risk neutral behavior), but also by considering risk-adverse agents which indeed account also for the "variance" of returns in their decisions. Moreover, it would be interesting to compare the performance of the investment strategies presented in Chapter 6 and Chapter 8 for real returns. Another interesting direction would be to compare the performance of the strategy based on a Genetic Algorithm against similar approaches from the area of Machine Learning like Genetic Programming, Neural Networks, and Reinforcement Learning.

Finally, it was shown in Chapter 9 how feedback processes based on a positive or negative experience may lead to the establishment of networks among agents. This feedback is considered a “social component” of the agents’ interaction and describes the establishment and reinforcement of relations among agents. The structures of the networks that appeared between agents and project initiators show that the interaction described above leads to the formation of common investment networks, where a positive experience with a given project initiator causes the agent to further “trust” him and to continue investing in his projects for a given number of time steps. It was shown that the agents’ budgets reach a stationary distribution after a certain number of time steps and present a power law distribution on the tail. Furthermore, the investment networks emerging from the model showed that the networks present some of the typical characteristics of real-life networks like a high clustering coefficient and short path length. However, the degree distribution of the investment networks does not follow a power-law behavior which is usually found in real-world networks, but rather a binomial distribution which is found in random networks.

For simplicity, we have assumed a random selection of failure or success, but we note that more elaborate economic assumptions, such as market dynamics based on supply and demand can be taken into account as well. It would be interesting to investigate the dynamics of the networks for a feedback mechanism which joins the previous experience of an agent with its further investment behavior. It would also be interesting to conduct a more detailed investigation of the role that the memory of an agent plays in determining the structure of the investment network. Finally, note that this model for the formation of common investment networks can be easily extended with regards to more complex cases. For example, it would be interesting to allow different projects to occur at the same time, as this might lead to competition between the agents for the chance to invest. Finally, instead of considering external incomes in the dynamics of the budget, it would be interesting to investigate an entry/exit system, where if an agent goes bankrupt, it is replaced by a new one.

ARCH/GARCH Processes

Time series with heteroskedasticity represent an interesting scenario for investors trying to find correlations in this type of time series. Because of this some researchers usually use them to simulate time series from the Stock Market, foreign Exchange Market and other financial time series [Podobnik et al., 2004]. One of the properties of heteroskedasticity is that after performing a regression analysis of the time series a non-linear plot of the residuals is observed. Typical stochastic process which generates time series with heteroskedasticity are the *ARCH* and the *GARCH* processes.

An *ARCH (Auto-Regressive Conditionally Heteroskedastic)* process is a a stochastic process with “nonconstant variance conditional on the past, but constant unconditional variance” [Engle, 1982].

An ARCH(p) process is defined by,

$$r(t) = \epsilon(t)\sigma(t) \tag{1}$$

$$\sigma(t)^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \, r(t-i)^2 \tag{2}$$

where $r(t)$ is a random variable with zero mean and variance $\sigma(t)^2$, $\epsilon_t \sim i.i.d.N(0,1)$, $\alpha_i \geq 0, \alpha_0 > 0$ and $p \in Z+$.

[Bera and Higgings, 1993] provide a survey on ARCH models, where some properties of these models and their applications are discussed.

A *GARCH (Generalized Auto-Regressive Conditionally Heteroskedastic)* process [Bollerslev, 1986], is a stochastic process with an autoregressive representation of the conditional variance and a moving average part. It is usually related to financial time series, because it presents distributions with fat tails and volatility clustering.

A GARCH(p,q) process is defined by,

$$r(t) = \epsilon(t)\sigma(t) \tag{3}$$

$$\sigma(t)^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \, r(t-i)^2 + \sum_{j=1}^{q} \beta_j \, \sigma(t-j)^2 \tag{4}$$

where $\alpha_i \geq 0, \beta_j \geq 0, \alpha_0 > 0$, $p, q \in Z+$ and $\epsilon_t \sim i.i.d.N(0,1)$.

For more information about autoregressive processes see [Mantegna and Stanley, 2000].

Equilibrium State of the Budget in Models based on Multiplicative Random Processes

In the following, we want to discuss more in detail some analytical aspects of the evolution and equilibrium state of the budget. In Section 2.2 it was shown that for the case of fixed income and fixed investment proportion, the distribution of the wealth over time reaches a stationary distribution presenting a power law in the tail. This was expected as a consequence of the results shown in Section 2.3.3. In the following, we try to show some theoretical aspects of the evolution of the budget over time and its equilibrium state.

To gain more insight into the dynamics of our investment model, we follow [Haan and Karandikar, 1989] to transform Eq. (2.9) into a stochastic differential equation by assuming that within a small time increment, $\triangle t$, the growth effects are reduced by this factor.

$$x(t+\triangle t) - x(t) = (x(t)\, r(t)\, q(t) + \, a(t))\, \triangle t \tag{5}$$

$$\frac{x(t+\triangle t) - x(t)}{\triangle t} = x(t)\, r(t)\, q(t) + a(t). \tag{6}$$

Taking the limit when $\triangle t \to 0$,

$$\lim_{\triangle t \to 0} \frac{x(t+\triangle t) - x(t)}{\triangle t} = \frac{\mathrm{d}x}{\mathrm{d}t} = x(t)\, r(t)\, q(t) + a(t). \tag{7}$$

If we assume that changes in $x(t)$ tend to be zero, $\mathrm{d}x = 0$, we find:

$$0 = x(t)\, r(t)\, q(t) + a(t). \tag{8}$$

From this previous, it can be seen that for a RoI with value of

$$r(t) = -\frac{a(t)}{x(t)\, q(t)}, \tag{9}$$

there would not be changes in the process for two consecutive time steps, $x(t) - x(t-1) = 0$. This means that for the process $x(t)$ to stay constant, the returns have to be negative and proportional to the income and the investment. Assuming that the agent has only control on the investment propotion $q(t)$ then, $x(t)$ may not change over time if:

$$r(t)\, q(t) = \frac{-a(t)}{x(t)}. \tag{10}$$

Note that the incomes are assumed to be $a(t) \geq 0$, then a constant budget over time can only be present for the following two cases: i) trivial case, if no incomes are present, $a(t) = 0$, and no investment is performed, $q(t) = 0$; and ii) if incomes are positive $a(t) > 0$, returns are negative $r(t) < 0$, and a proper $q(t)$ is used to balance the increments and decrements in budget due to incomes and returns. The latter case is visualized in Fig. 1, we show the logarithm of the budget instead of the budget in order to allow a graphical visualization of the possible combination of RoI $r(t)\,q(t)$ and incomes $a(t)$ in Eq.(10), leading to a constant budget, $x(t) - x(t+1) = 0$.

Based on visual impression, for $x(t) > 0.1$, as the budget increases (mainly because of the successive addition of incomes, $a(t)$), the agent should decrease the proportion of investment, $q(t)$, in order to have a constant budget. On the other hand, if $x(t)$ is small, $x(t) < 0.1$, an equilibrium can only be achieved if the agent receives small incomes no mattering the RoI.

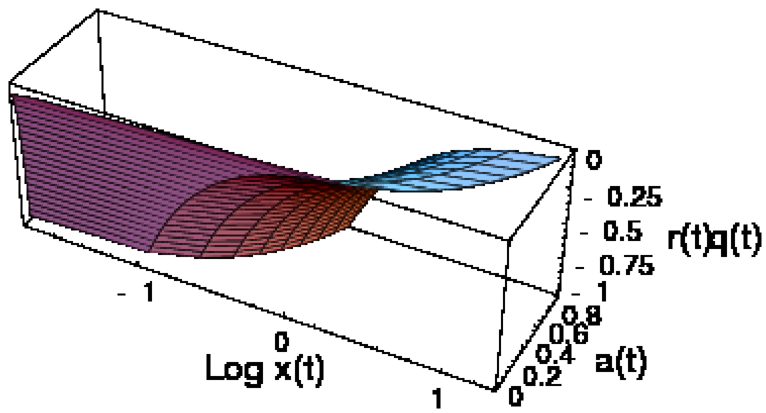


Figure 1.: Payoff $r\,q$ and income a for which the budget in Eq.(2.9) is stable.

However, the investor does not want to remain with the same wealth over time, on the contrary, the investor would like to increase always the budget if possible. Thus, from Eq. (10), it can be seen that even receiving large negative return, the investor can always increase the wealth over time if the proportion of investment is updated as follows:

$$q(t) \leq -\frac{a}{x(t)\,r^{-}_{max}}. \tag{11}$$

where r^{-}_{max} is the maximal negative return that the investor can receive, which in most of our examples is $r^{-}_{max} = -1$. Thus, given $x(t)$, $a(t)$ and the most negative possible RoIs, the values of $q(t)$ that lead to a no-decrease of wealth, are any of those leading to values of x above the plane in Fig. 1. However, this previous leads to the uninteresting situation that as $x(t) \to \infty$, $q(t) \to 0$, which is another reason why in order to enforce the dynamics in most of our simulations the minimum proportion of investment value has to be larger than zero, see Section 2.2.

Periodic Returns and Total Budget at the End of a Cycle

For the sake of clarity, in the following, it is demonstrated that for periodic returns as those described in Eq. (6.2), the budget at the end of the cycle is not being influenced by the phase of the returns. Firstly, the returns here considered are as described in Eq. (6.2) for $w = \frac{2\pi}{T} + \phi$. Fig. 2 (left) shows the returns for different phase values: $\phi = \{0, \pi/2, \pi\}$. Secondly, Fig. 2 (right) shows the dynamics of the investment strategies here considered: Constant Proportion (CP) Section 6.4.1, Ramp-Rectangle (RR) strategy Section 6.4.2 and the Square-Wave strategy (SW) that is a variant of the RR strategy.

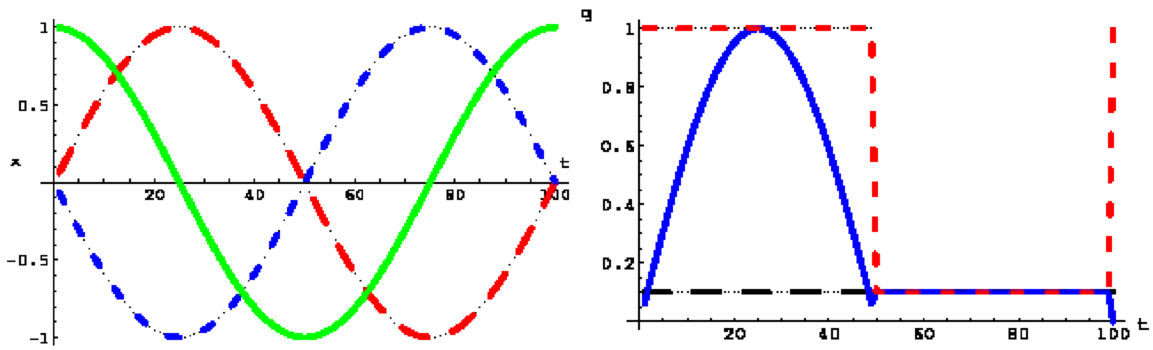


Figure 2.: (left) Return on Investment, Eq. (6.2), with $w = \frac{2\pi}{T} + \phi$ for different phase values: $\phi = \{0, \pi/2, \pi\}$. (right) Investment strategies: (long-dashed) Constant Proportion (CP), (solid) Ramp-Rectangle (RR) and (short-dashed) Square-Wave (SW).

Fig. 3 show the evolution of the budget over time for periodic returns showed in Fig. 2 (left) and the investment strategies showed in Fig. 2 (right). It can be seen in Fig. 3 (top) that for the Ramp-Rectangle strategy (RR), which invests always a fixed investment, the budget at the end of the cylce is the same as the initial budget. On the other hand, Fig. 3 (middle) shows that for the Ramp-Rectangle strategy, which invests always proportionally to the return, the budget at the end of the cylce is larger than the initial budget, leading to profits over 30 units, however, Fig. 3 (bottom) shows that for the Square-Wave strategy, which invests all or nothing depending on the sign of the expected return, the budget is much larger than for the other strategies outperforming the other two strategies.

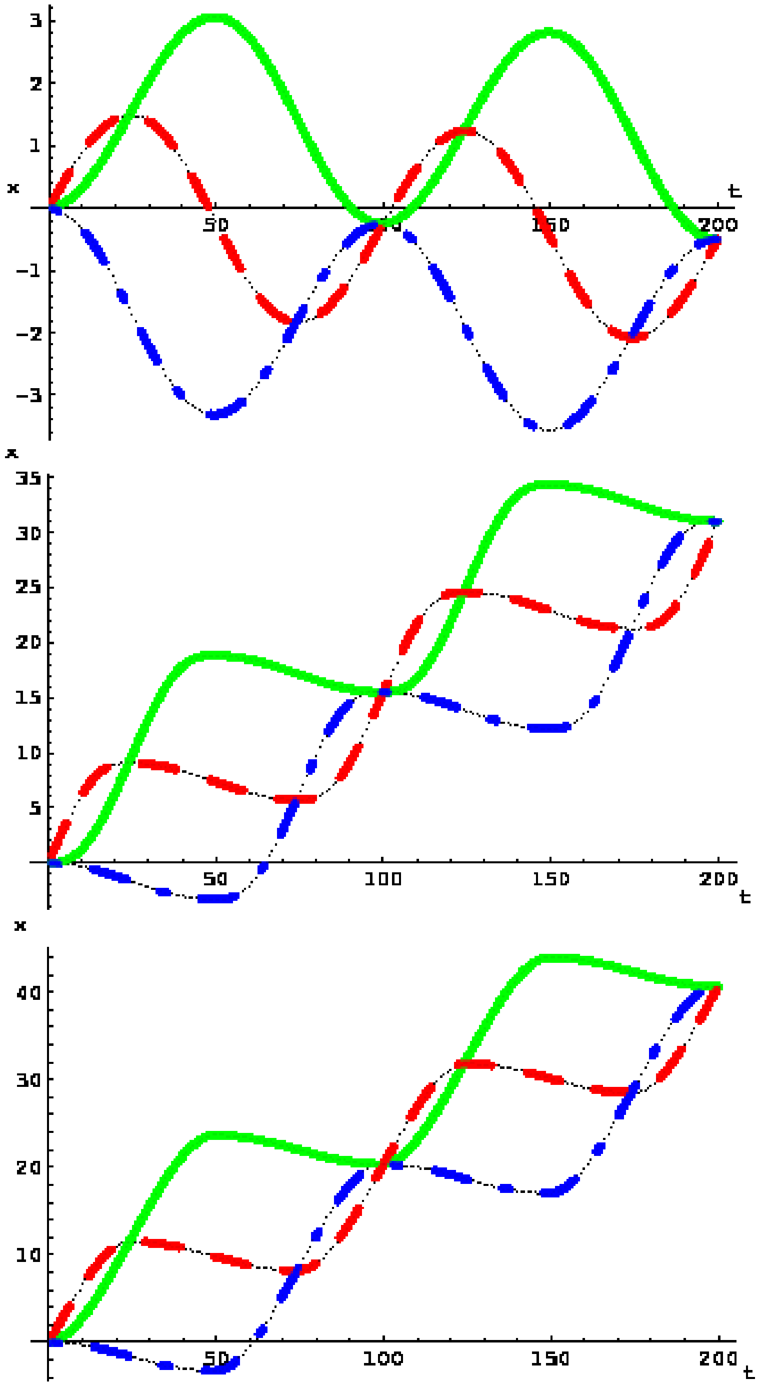


Figure 3.: Evolution of the budget for returns as in Fig. 2 (left) and for strategies as in Fig. 2 (right): (top) Constant Proportion, (middle) Ramp-Rectangle and (bottom) Square-Wave.

Utility of Wealth and Fixed Investment

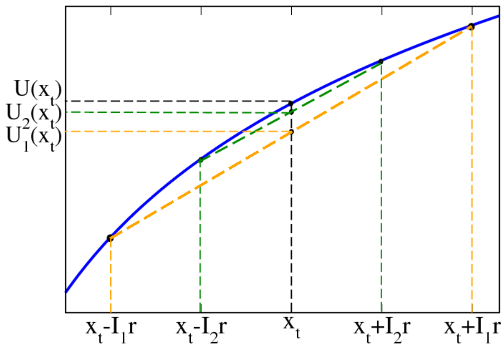
Usually, researchers use utility theory to express individuals different preferences among possible situations, goods, choices, etc. In agent theory, utility functions are used for agents to take decisions in different types of contexts. In general, the concept of utility can be explained as the way an agent ranks all possible situations, choices or goods from the less to the most desirable, in which the most desirable ones have a higher utility than the less desirable ones. By this means, agent's preferences can be expressed via a utility function, which assigns to the possible combination of choices a single value which reflects how much the agent would like to receive them. Note that because of the non-uniqueness of utility measures, when comparing the utilities between for example two possible goods, it does not matter how much is one preferred than the other, in other words, the only fact that matters is that one choice is being preferred more than the other. Thus, we can have an agent using the utility function $U(X, Y)$ to rank her preferences between two goods, where X and Y represent the quantities of the two goods. Also note that the utility functions can have other kind of arguments, for example the utility an agent gives to money $U(x)$, or agent's consumption choice for different time periods can be described with the utility function $U(C_1, C_2)$, with C_1 the consumption in the first period and in C_2 in the second period. For further details regarding utility theory in decision problems refer to [Nicholson, 1992, chap. 3] and [Russell and Norvig, 1995, chap. 14].

In the rest of this appendix, we describe the relationship between agent's utility function of wealth and her proportion of investment. For simplicity, we refer indistinctly to a variable at time t with a functional or a subindex, for example the wealth of an agent at time step t may be refered as $x(t)$ or x_t.

Consider an agent who wants to invest his money in an investment intrument that yields a RoI r. Assume that he can choose between two possible proportions of investment q_1 and q_2, where $q_1 > q_2$, the agent should decide between investing $I_1 = x_t q_1$ or $I_2 = x_t q_2$, where x_t is the actual budget. Fig. 4 (left) shows the utilities U_1 and U_2 for I_1 and I_2 respectively, for an agent with a concave utiliy function of wealth. Note that $U_1(x_t) < U_2(x_t)$, which means that the agent would rather choose the proportion of investment q_2 because this leads to higher utility. However, he would rather like not to invest at all, because the utility of the actual wealth is higher, $U(x_t) > U_2(x_t) > U_1(x_t)$.

Thus, for random returns, an agent with a concave utility function of wealth, $\frac{d^2U}{dx^2} < 0$, will always choose a small value of q, leading to smaller variability of return and if possible it may not even want to invest or risk at all, i.e. $q = 0$. This kind of agent is called a risk-averse agent. On the other hand, a risk-neutral agent will choose either of the two proportions of investment and his behavior can be represented using a linear utility function of wealth, whereas a risk-seeking agent will choose q_2, i.e. the investment with larger variability of return and his behavior can be represented using a convex utility function of wealth, $\frac{d^2U}{dx^2} > 0$.

Fig. 4 (right) shows an example of these different kind of behaviors, where x_t is the actual budget of the agent. In this example, a risk-averse agent is represented with the utility function $U_a(x) = b \log(x)$ where $b = 1/0.3$, note that $\frac{d^2U_a}{dx^2} = \frac{-b}{x^2}$, i.e. $\frac{d^2U_a}{dx^2} < 0$.



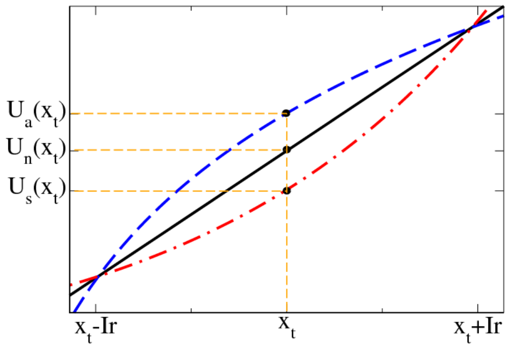


Figure 4.: Utility function of wealth for: (left) an agent with a concave utility function and budget x_t, two possible investments I_1 and I_2 for a random return r. In this case, if $I_1 > I_2$, then $U_1(x_t) < U_2(x_t)$; (right) Utility functions of wealth for: (dashed) risk-averse, (solid) risk-neutral and (dashed-point) risk-seeking agent.

The risk-neutral agent has an utility function $U_n(x) = x$, and the risk-seeking agent has an utility function $U_s(x) = \exp(c\,x)$ where $c = 1/b$, note that $\frac{d^2U_a}{dx^2} = c^2 \exp(c\,x)$, i.e. $\frac{d^2U_s}{dx^2} > 0$.

Note that if the the odds are even (fair bets) and the agent has the possibility to double his investment or to lose it, then a risk-averse agent will always refuse to invest, whereas a risk-neutral agent will be indifferent in investing or not, and a risk-seeking agent will always accept to invest. Thus, given an agent's utility function of wealth, we can find out if agent's behaviour is risk-averse, risk-neutral or risk-seeking, as well as if he'll like to invest in a double or nothing Return on Investment (RoI). However, it would be useful to quantify the degree of risk-aversion of an agent. Pratt [1964] showed two different measures of risk aversion known as ARA (absolute risk aversion) and RRA (relative risk aversion) which quantify how averse an agent is when making decisions in risky situations. These measures of risk-aversion are based on agent's utility function of wealth and can be interpreted as a measure of agent's liking to be insure or to be propense to the risk. Furthermore, these measures are also used to obtain the *risk-premium*, which in the case of a risk-averse agent is the amount the agent will be willing to pay in order to avoid the bet, i.e. to avoid performing the investment. In the following, we discuss briefly these measures of risk-aversion and the calculation of the risk-premium as they are described by Pratt [1964].

Let us assume that at every time step, the agent has to choose between receiving: (i) a random amount $z = I\,r$, where, I is the amount of wealth being invested and r is a random variable which represents the RoI; and (ii) a non-random amount $E[z] - \pi(x, z)$, where $E[z]$ denotes the expected value of the random amount z and $\pi(x, z)$ is the risk-premium. Thus, if the market has equally probable negative and positive returns, then the agent would be indifferent between receiving the random amount z and the non-random amount $E[z] - \pi(x, z)$. For example, if we assume that the returns are binomial distributed, i.e. $r \in \{-1, +1\}$, this means that the agent would be indifferent between risking his investment in a market which doubles or takes agent's investment; and receiving for sure the non-random amount $-\pi(x, z)$ (note that in this case $E[z] = 0$). Then, if an agent is risk-averse, $\pi(x, z) > 0$, this means that he would be willing to pay at most the amount $\pi(x, z)$ in order to avoid taking the risk (performing the investment), which in other words means that this agent would be willing to receive at least the amount $\pi(x, z)$ to be convinced to perform the investment. On the oher hand, a risk-seeking agent, $\pi(x, z) < 0$, would be willing to receive at least the amount $\pi(x, z)$ to neglect the investment, which conversely means that he would be willing to pay at most $\pi(x, z)$ to have the opportunity to perform the investment. These previous situations may sound strange to the reader at first, however, these can be explained in other terms. For example, the situation where an agent may be willing to pay in order to avoid the risk can be expressed in terms of insurance premiums and credit interests. Consider an agent who is afraid of losing some ownership, he may me willing to pay for some protection against the possible loss. In the same situation, an agent may be convinced to give a credit if it is promised to receive also some interests. This rewards the risk of not having his money back.

On the other hand, the situation where an agent may receive money to neglect taking the risk can be expressed in terms of employment/unemployment. Consider an agent that wants to leave firm A and would like to try luck in firm B which gives higher incomes and promises a nicer environment. Firm A in this case would like to convince the agent not to resign by raising his salary. Thus, the agent would be willing to receive at least some raise amount in order to neglect taking the risk accepting the employment offer by firm B. Conversely, if the agent has a fixed-term contract with firm A, he would like to pay firm A the corresponding penalty because of resigning before the expiration of his employment contract (for example when being part of a project).

Now, let us focus in the case where the agent invests a fraction q of its wealth x, i.e. the agent invests the amount $I = q\,x$. For completeness, we start describing the *absolute risk-aversion measure (ARA)* which is defined for fixed investments $I = const$ and is calculated as follows:

$$ARA(x) = -\frac{U''(x)}{U'(x)}, \tag{12}$$

where $U''(x) = \frac{d^2U}{dx^2}$ and $U'(x) = \frac{dU}{dx}$. The *risk-premium*, $\pi(x, z)$ is proportional to the measure of absolute risk-aversion $ARA(x)$ and can be calculated as follows:

$$\pi(x, z) = \frac{1}{2}\,\sigma_z^2\,ARA(x) + o(\sigma_z^2), \tag{13}$$

where σ_z^2 is the variance of the random variable $z = I\,r$, and $o(\cdot)$ is a term of smaller order.

Note that if we assume that agent's utility function of wealth is defined as

$$U(x) = -\exp{(b\,x)}\ ; \quad b < 0, \tag{14}$$

then the ARA measure of aversion is constant, $ARA(x) = -b$, which means that the agent has a constant absolute risk-aversion (CARA) for all his levels of wealth.

To clarify this fact, consider an agent with an utility function of wealth Eq. 14 with $b = -1$. This means that the agent has a constant absolute risk aversion of $ARA(x) = 1$. If the agent has a budget of $x = 1$ and the possibility to invest the fixed amount $I = 0.1$, then his risk-premium would be $\pi(x, z) = 0.005$. This means that he would be willing to pay at most the amount 0.005 to avoid taking the risk. Note that the same amount would be payed by the agent if he has a budget of $x = 10$ or $x = 100$. Moreover, if the agent increases his risk aversion using $b = -2$ and $b = -5$ in Eq. 14, then the risk premiums would be $\pi(x, z) = 0.01$ and $\pi(x, z) = 0.025$ respectively, i.e. quantities that are proportional to the measure of absolute risk aversion $ARA(x)$.

Now, consider the case where the agent is investing not a fixed amount of money but a fraction q of his wealth x, i.e. $I = q\,x$. In this case, if the market has equally probable negative and positive returns, the agent would be indifferent between receiving the random amount $x\,z$ and the non-random amount $E[x\,z] - x\,\pi^\star(x, x\,z)$, where $E[x\,z]$ denotes the expected value of the random amount and $\pi^\star(x, x\,z)$ is the relative risk-premium, which is calculated similarly to Eq. (13) by:

$$\pi^\star(x, x\,z) = \frac{1}{2}\sigma_z^2 RRA(x) + o(\sigma_z^2), \tag{15}$$

where σ_z^2 is the standard deviation of z and the measure of *relative risk aversion (RRA)* is calculated as follows:

$$RRA(x) = \frac{-x\,U''(x)}{U'(x)}. \tag{16}$$

Consequently, the *relative riskless amount*

$$\rho^\star(x, x\,z) = E[x\,z] - x\,\pi^\star(x, x\,z), \tag{17}$$

is the amount that makes an agent to be indifferent in choosing between the random and the non-random amount. Note that the sign of $\rho^\star(x, x\,z)$ denotes if the agent has to pay (negative sign) or to receive (positive sign) the corresponding amount $\rho^\star$.

Thus, a quadratic utility function of wealth exhibits a decreasing $RRA(x)$, for example for $U(x) = a + b\,x + c\,x^2$ we have $RRA(x) = \frac{b}{b+2\,c\,x} - 1$, whereas an exponential exhibits an increasing $RRA(x)$ for example Eq. 14 has a $RRA(x) = -b\,x$, and logarithmic and linear functions exhibit a *constant relative risk aversion (CRRA)*, for example $U(x) = b\,logx$ and $U(x) = x$ with $RRA(x) = 1$, and $RRA(x) = 0$ respectively.

However, the following power utility function of wealth is of special interest for us:

$$U(x) = x^k \tag{18}$$

Interestingly, this utility function exhibits a CRRA which value depends only on the value of k,

$$RRA(x) = 1 - k \tag{19}$$

By this means, it is possible to relate k with agent's attitude towards risk, for example if we assume $k \in [-1, 3]$, we find that a risk-averse agent has a $RRA(x) > 0$ with $k \in [-1, 1)$. The smaller k the more risk-averse the agent would be. A risk-neutral agent has a $RRA(x) = 0$ with $k = 1$ and a risk-seeking agent has $RRA(x) < 0$ with $k \in (1, 3]$. The larger k the more risk-seeking the agent would be. These different attitudes toward risk are also shown in Fig. 5, where the utility functions of wealth Eq. (18) is drawn for different k values in a log-log plot.

Consider an agent with the utility function of wealth in Eq. (18), an initial wealth of $x = 1$ and decides to invest a fraction of $q = 0.1$ of his wealth in an investment instrument with binomial returns $r \in \{-1, 1\}$. This means that the agent may lose or gain $q\,x$ with equal probability, where the standard deviation of $z = q\,r$, is $\sigma_z^2 = q^2$. Replacing Eq. (19) in Eq. (15), we can express the relative risk-premium for binomial returns as follows:

$$\pi^\star(x, x\,z) = \frac{1}{2}\,q^2\,(1 - k) \tag{20}$$

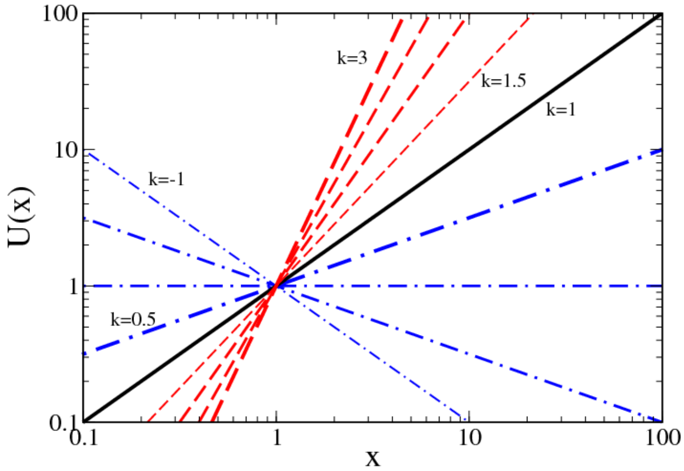


Figure 5.: Utility function of wealth, Eq. 18, for different values of k. For agents that are: risk-averse (dot-dashed) $k \in [-1, 1)$, risk-neutral (solid) $k = 1$, and risk-seeking (dashed) $k \in (1, 3]$.

Thus, if the agent has $k = -1$ this means that the agent is risk-averse with $RRA(x) = 2$, then his relative risk-premium is $\pi^{\star}(x, x\,z) = 0.01$, i.e. $\rho(x, x\,z) = -0.01$. This means that he may be willing to pay at most the amount of 0.01 in order to avoid performing the investment. Now, if the agent has $x = 10$ and the same proportion of investment $q = 0.1$, then his relative risk-preimum would be again $\pi^{\star}(x, x\,z) = 0.01$, i.e. $\rho(x, x\,z) = -0.1$. In this case he may be willing to pay at most the amount of 0.1 to avoid the investment.

This linear relationship between the relative riskless amount, $\rho(x, x\,z)$, and the budget x is shown in Fig. 6 (left) for different k values, in which we assume that the agent has a proportion of investment $q = 0.1$. As it was mentioned before, for this case the agent would be indifferent between risking the amount $q\,x$ in an investment option with binomial returns and receiving for sure the amount $\rho(x, x\,z)$.

However, if the agent has $x = 10$, $q = 0.1$ and utility Eq. (18) with $k = 0$, he is risking the amount $I = 1$ and he would be willing to pay no more than 0.05 to avoid the investment, which is much less than the investment amount he would be investing. On the other hand, if the agent has $q = 0.99$, the agent now may be investing almost his whole capital, $I = 9.9$, in the investment instrument, in which case he would be willing to pay no more than 4.9 to avoid the investment. It is clear from Fig. 6 (right) that the relative risk-premium $\pi^{\star}(x, x\,z)$, Eq. 20, is not linearly proportional to the proportion of investment q for all values of k, excluding $k = 1$. This non-linear relationship may be intuitive realistic. For example, for a risk-averse agent the larger q the larger the amount that he would be willing to pay in order to avoid the investment.

However, one may be interested in having a linear relationship between $\pi^{\star}(x, x\,z)$ and q. To our knowledge such a linear relationship between the proportion of investment and the relative risk premium has not been proposed yet, thus, we proceed to find it by including the proportion of investment q in the utility function of wealth. Assume that the exponent

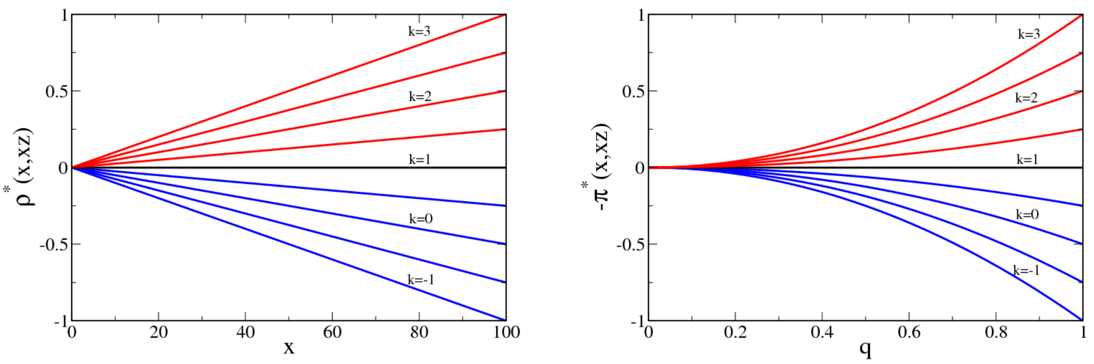


Figure 6.: Relation between: (left) the relative riskless amount $\rho(x, x\,z)$, Eq. (17), and the budget x, for $q = 0.1$ and different k; (right) the relative risk-premium $\pi^\star(x, x\,z)$, Eq. (20) and the proportion of investment q for different k values.

k in Eq. (18) is now defined by,

$$k = 1 + \frac{2\,q\,\ell}{\sigma_z^2}, \tag{21}$$

where ℓ describes the risk-propensity, $\ell \in [-1, +1]$. If an agent chooses $\ell = -1$, this means that the agent is completely risk-averse, for $\ell = 0$ means the agent is risk-neutral, whereas for $\ell = 1$ the agent is completely risk-seeking. Then, the utility function of wealth can be described by the risk-propensity ℓ and the proportion of investment q, given the standard deviation of the returns, σ_z^2. Fig. 7 shows (for visibility reasons) the natural logarithm of this function for different values of ℓ and for two different fixed proportion of investment values $q = 0.1$ (left) and $q = 0.5$ (right). Observe that the range of utility values decreases as q increases. For this case we assumed binomial returns, i.e. $\sigma_z^2 = q^2$, and we observe that as in Fig. 5, there are power law relationships between the utility and the budget for different parameter values.

Now, by substituting Eq. (21) in Eq. (18), the resulting utility function has the following first and second derivatives:

$$U'(x) = \left(1 + \frac{(2\,q\,\ell)}{\sigma_z^2}\right) x^{\left(\frac{2\,q\,\ell}{\sigma_z^2}\right)} \tag{22}$$

$$U''(x) = \frac{2\,q\,\ell\left(1 + \frac{(2\,q\,\ell)}{\sigma_z^2}\right)}{\sigma_z^2}\, x^{\left(-1+\frac{2\,q\,\ell}{\sigma_z^2}\right)} \tag{23}$$

And using Eq. (12) and (16), the ARA and RRA are respectively:

$$ARA(x) = -\frac{2\,q\,\ell}{x\,\sigma_z^2} \tag{24}$$

$$RRA(x) = -\frac{2\,q\,\ell}{\sigma_z^2} \tag{25}$$

Interestingly, if Eq. (25) is substituted in Eq. (15) we find that the relative risk premium is

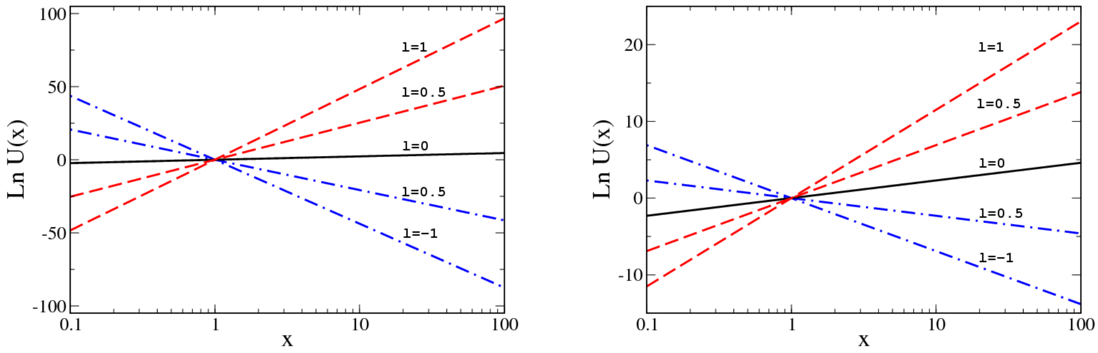


Figure 7.: Utility function of wealth Eq. (18) with k defined as in Eq. (21) for fixed proportions of investment: (left) $q = 0.1$ and (right) $q = 0.5$. For binomial returns and different risk-propensity ℓ.

given then by,

$$\pi^{\star}(x, x\, z) = -q\, \ell. \tag{26}$$

Fig. 8 shows the linear relationship between the relative risk premium and the proportion of investment q for different risk-propensity ℓ.

For example, if the agent has budget $x = 10$, and proportion of investment $q = 0.1$, i.e. $I = 1$, if we assume that the agent has $\ell = -0.9$, this means that $\rho = -0.9$ and he may be willing to pay the amount of 0.9 in order to avoid the investment. The previous may sound irrealistic, but consider the case in which the agent wants to be really sure that he will have a small amount of money for the next time step, in this example at least 0.1. Obviously, the case where agent has $\ell = -1$ may imply that the agent wants to endow or donate the amount 1, giving no chance to random amounts. On the other hand, assuming the previous parameter values and a risk-propensity $\ell = 0.9$, the agent now wants to invest and he would be willing to neglect the investment only if he receives an amount larger than $\rho = 0.9$ Finally, if the agent has $\ell = 1$, this may imply that the agent is 100% sure that the investment will yield very large profits and because of this, he lets no chance to non-random amounts.

Note that this previous is true for all types of returns because the random properties of the returns r are taken into account in the utility function only, specifically in the standard deviation of $z = q\, r$, via σ_z^2. In the previous presented examples, we considered binomial returns, $r(t) \in \{-1, 1\}$ with equal probability of ocurrance, which leads to $\sigma_z^2 = q^2$. If the returns are drawn from other distributions, we may want to calculate the value of σ_z^2 for the utility function. For example, if returns are randomly drawn from a uniform distribution $r \sim U(-1, 1)$, we have:

$$\sigma_z^2 = \int_{-q}^{q} z^2\, dz = \frac{2\, q^3}{3}. \tag{27}$$

Thus, in terms of utility theory, the investigations done in Chapter 3 showed results for an agent that has a constant proportion of investment q. In other words, this means that the agent is characterized by a power utility function of wealth as in Eq. (18) with k defined as in (21) with risk-propensity $\ell = 0$, i.e. in this case the agent is risk-neutral with utility

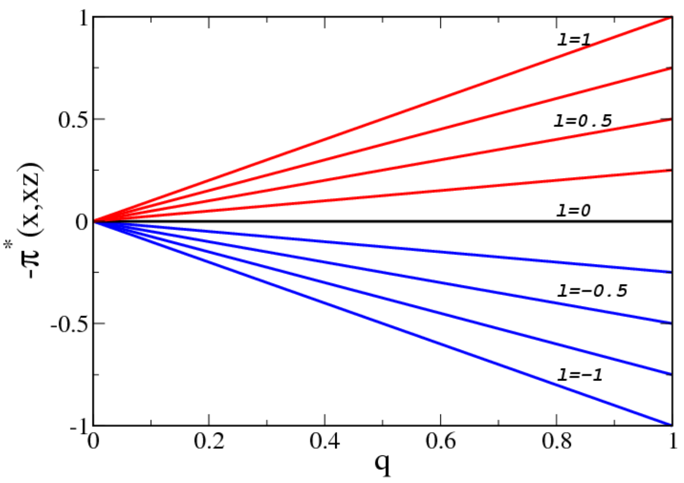


Figure 8.: Relative risk-premium Eq. (26) is linearly proportional to the proportion of investment q for any value of the risk-propensity ℓ in the range $[-1, 1]$. For an agent using the utility function of wealth in Eq. (18) with k as in Eq. (21).

function of wealth $U(x) = x$. However, note that these previous explanations imply that the investigations in this PhD thesis can be extended to include utility theory without losing generality.

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Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 14.07.2008 Jesús Emeterio Navarro-Barrientos